Classifying Descents According to Parity

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Abstract

In this paper we refine the well-known permutation statistic "descent" by fixing parity of (exactly) one of the descent's numbers. We provide explicit formulas for the distribution of these (four) new statistics. We use certain differential operators to obtain the formulas. Moreover, we discuss connection of our new statistics to the Genocchi numbers. We also provide bijective proofs of some of our results.

Keywords: permutation statistics, descents, parity, distribution, bijection

1 Introduction

The theory of permutation statistics has a long history and has grown rapidly in the last few decades. The number of *descents* in a permutation π, denoted by $des(\pi)$, is a classical statistic. The *descent set* of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is the set of indices i for which $\pi_i > \pi_{i+1}$. This statistics was first studied by MacMahon [10] almost a hundred years ago, and it still plays an important role in the field.

Eulerian numbers $A(n,k)$ count permutations in the symmetric group $\mathcal{S}_n$ with k descents and they are the coefficients of the *Eulerian polynomials* $A_n(t)$ defined by $A_n(t) = \sum_{\pi\in\mathcal{S}_n} t^{1+des(\pi)}$. The Eulerian polynomials satisfy the identity

$$\sum_{k\geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

For more properties of the Eulerian polynomials see [3].

By definition, $\pi_i\pi_{i+1}$ is a descent in a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ if and only if $\pi_i > \pi_{i+1}$. In this paper we consider refinement of the notion of a descent by fixing parity of either the first or the second number (but not both!) of a descent. For example, if we require that the second number of a descent must be even, then we do not consider at all descents ending with an odd number. Thus we introduce four new permutation statistics.

We provide exact formulas for the distribution of our new statistics. That is, we give two formulas (for even and odd n) for the number of n-permutations having exactly k descents of a chosen type. These formulas for "beginning with an even number" (resp. "ending with an even number," "beginning with an odd number," and "ending with an odd number") case can be found in Section 3 (resp. 4, 5, and 6). In the first three cases above, we use certain differential operators to derive explicit answers; we then use trivial bijections of the symmetric group to itself to proceed with the fourth case.

In Section 7 we link our descents to *patterns* in permutations (see [2] for an introduction to the subject). Using certain patterns, which are an alternative notation for our descents, we provide equivalent definitions of the *Genocchi numbers* both on even and odd permutations. In Section 8 we state bijective proofs for certain results on descents according to parity and, in Section 9, we discuss patterns in which generalized parity considerations are taken into account.

2 Definitions and notations

Let $\mathcal{S}_n$ denote the set of permutations of $\{1, 2, \ldots, n\}$. Let $E = \{0, 2, 4, \ldots\}$ and $O = \{1, 3, 5, \ldots\}$ denote the set of even and odd numbers respectively. Given $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$, we define the following, where $\chi(\sigma_1 \in X)$ is 1 if σ_1 is of type X, and it is 0 otherwise.

- $\overleftarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \&\ \sigma_i \in X\}$ and $\overleftarrow{des}_X(\sigma) = |\overleftarrow{Des}_X(\sigma)|$ for $X \in \{E, O\}$;
- $\overrightarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \&\ \sigma_{i+1} \in X\}$ and $\overrightarrow{des}_X(\sigma) = |\overrightarrow{Des}_X(\sigma)|$ for $X \in \{E, O\}$;
- $R_n(x) = \sum_{\sigma\in S_n} x^{\overleftarrow{des}_E(\sigma)}$ and $P_n(x, z) = \sum_{\sigma\in S_n} x^{\overrightarrow{des}_E(\sigma)} z^{\chi(\sigma_1\in E)}$;
- $M_n(x) = \sum_{\sigma\in S_n} x^{\overleftarrow{des}_O(\sigma)}$ and $Q_n(x, z) = \sum_{\sigma\in S_n} x^{\overrightarrow{des}_O(\sigma)} z^{\chi(\sigma_1\in O)}$;
- $R_n(x) = \sum_{k=0}^{n} R_{k,n}x^k$ and $P_n(x, z) = \sum_{k=0}^{n}\sum_{j=0}^{1} P_{j,k,n}z^j x^k$;
- $M_n(x) = \sum_{k=0}^{n} M_{k,n}x^k$ and $Q_n(x, z) = \sum_{k=0}^{n}\sum_{j=0}^{1} Q_{j,k,n}z^j x^k$.

Thus our goal in this paper is to study the coefficients $R_{k,n}$, $P_{j,k,n}$, $M_{k,n}$, and $Q_{j,k,n}$ of the polynomials $R_n(x)$, $P_n(x)$, $M_n(x)$, and $Q_n(x)$ respectively.

Given any permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n$, we label the possible positions of where we can insert $n+1$ to get a permutation in $\mathcal{S}_{n+1}$ from left to right with 0 to n, i.e., inserting $n + 1$ in position 0 means that we insert $n + 1$ at the start of σ and for $i \geq 1$, inserting $n + 1$ in position i means we insert $n + 1$ immediately after σ_i. In such a situation, we let $\sigma^{(i)}$ denote the permutation of $\mathcal{S}_{n+1}$ that results by inserting $n + 1$ in position i.

Let $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n)$ denote the *complement* of σ. Clearly, if n is odd, then, for all i, σ_i and $n + 1 - \sigma_i$ have the same parity, whereas they have opposite parity if n is even. The *reverse* of σ is the permutation $\sigma^r = \sigma_n\sigma_{n-1}\cdots\sigma_1$.

3 Beginning with an even number: properties of $R_n(x)$

Let Δ_{2n} be the operation which sends x^k to $kx^{k-1} + (2n+1-k)x^k$ and Γ_{2n+1} be the operator that sends x^k to $(k+1)x^k + (2n+1-k)x^{k+1}$. Then we have the following.

Theorem 1. *The polynomials $\{R_n(x)\}_{n\geq 1}$ satisfy the following recursions.*

1. $R_1(x) = 1$ *and* $R_2(x) = 1 + x$,
2. $R_{2n+1}(x) = \Delta_{2n}(R_{2n}(x))$ *for* $n \geq 1$, *and*
3. $R_{2n+2}(x) = \Gamma_{2n+1}(R_{2n+1}(x))$ *for* $n \geq 1$.

Proof. Part 1 is easy to verify by direct computation.

For part 2, suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{S}_{2n}$ and $\overleftarrow{des}_E(\sigma) = k$. It is then easy to see that if we insert $2n+1$ in position i where $i \in \overleftarrow{Des}_E(\sigma)$, then $\overleftarrow{des}_E(\sigma^{(i)}) = k-1$. However, if we insert $2n+1$ in position i where $i \notin \overleftarrow{Des}_E(\sigma)$, then $\overleftarrow{des}_E(\sigma^{(i)}) = k$. Thus $\{\sigma^{(i)} : i = 0, \ldots, 2n\}$ gives a contribution of $kx^{k-1} + (2n+1-k)x^k$ to $R_{2n+1}(x)$.

For part 3, suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1} \in \mathcal{S}_{2n+1}$ and $\overleftarrow{des}_E(\sigma) = k$. It is then easy to see that if we insert $2n+2$ in position i where $i \in \overleftarrow{Des}_E(\sigma)$ or $i = 2n+1$, then $\overleftarrow{des}_E(\sigma^{(i)}) = k$. Similarly if we insert $2n+2$ in position i where $i \notin \overleftarrow{Des}_E(\sigma) \cup \{2n+1\}$, then $\overleftarrow{des}_E(\sigma^{(i)}) = k+1$. Thus $\{\sigma^{(i)} : i = 0, \ldots, 2n+1\}$ gives a contribution of $(k+1)x^k + (2n+1-k)x^{k+1}$ to $R_{2n+2}(x)$. $\square$

We can express Theorem 1 in terms of differential operators:

Corollary 1. *The polynomials $\{R_n(x)\}_{n\geq 1}$ are given by the following*

1. $R_1(x) = 1$, $R_2(x) = 1 + x$, *and for* $n \geq 1$,
2. $R_{2n+1}(x) = (1-x)\frac{d}{dx}R_{2n}(x) + (1+2n)R_{2n}(x)$ *and*
3. $R_{2n+2}(x) = x(1-x)\frac{d}{dx}R_{2n+1}(x) + (1 + x(1+2n))R_{2n+1}(x)$.

This given, we can easily compute some initial values of $R_n(x)$.

$R_1(x) = 1.$

$R_2(x) = 1 + x.$

$R_3(x) = 4 + 2x.$

$R_4(x) = 4 + 16x + 4x^2.$

$R_5(x) = 36 + 72x + 12x^2.$

$R_6(x) = 36 + 324x + 324x^2 + 36x^3.$

$R_7(x) = 576 + 2592x + 1728x^2 + 144x^3.$

$R_8(x) = 576 + 9216x + 20736x^2 + 9216x^3 + 576x^4.$

Theorem 2. *We have* $R_{0,2n} = R_{n,2n} = (n!)^2$.

Proof. It is easy to see that the theorem holds for $n = 1$.

Now suppose that $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ is such that $\overleftarrow{des}_E(\sigma) = 0$. Then we can factor any such permutation into blocks by reading the permutation from left to right and cutting after each odd number. For example if $\sigma = 1\ 2\ 4\ 5\ 3\ 6\ 7\ 9\ 8\ 10$, then the blocks of σ would be 1, 2 4 5, 3, 6 7, 9, and 8 10. Since $\overleftarrow{des}_E(\sigma) = 0$, there must be a block of even numbers at the end which must contain the number $2n$ and which are arranged in increasing order. We call this final block the n-th block. Every other block must end with an odd number $2k+1$ which can be preceded by any subset of even numbers which are less than $2k+1$ arranged in increasing order. We call such a block the k-th block. It is then easy to see that there are $n!$ ways to put the even numbers $2, 4, \ldots, 2n$ into the blocks. That is, $2n$ must go in the n-th block since if we place $2n$ anywhere but at the end of the permutation, it would contribute to $\overleftarrow{Des}_E(\sigma)$. Then $2(n-1)$ can either go in block $n-1$ or block n. More generally, $2(n-k)$ can go in any blocks $(n-k), \ldots, n$. Once we have arranged the even numbers into blocks, it is easy to see that we can arrange blocks $0, \ldots, n-1$ in any order and still get a permutation σ with $\overleftarrow{des}_E(\sigma) = 0$. It thus follows that there are $(n!)^2$ such permutations. Hence $R_{0,2n} = (n!)^2$.

Now suppose that $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ is such that $\overleftarrow{des}_E(\sigma) = n$. Then, as above, we can factor any such permutation into blocks by reading the permutation from left to right and cutting after each odd number. For example if $\sigma = 4\ 2\ 1\ 5\ 6\ 3\ 10\ 8\ 7\ 9$ then the blocks of σ would be 4 2 1, 5, 6 3, 10 8 7, and 9. Since $\overleftarrow{des}_E(\sigma) = n$, each even number must start a descent and hence, unlike the case where $\overleftarrow{des}_E(\sigma) = 0$, there can be no even numbers at the end. Thus the n-th block must be empty. It is also easy to see that if $0 \leq k \leq n-1$ and there are even numbers in the k-th block, i.e. the block that ends with $2k+1$, then those numbers must all be greater than $2k+1$ and they must be arranged in decreasing order.

It is then easy to see that there are $n!$ ways to put the even numbers $2, 4, \ldots, 2n$ into blocks. That is, $2n$ may go in any of the blocks 0 through $n-1$, $2(n-1)$ can go in any of the blocks 0 through $n-2$, etc. After we have partitioned the even numbers into their respective blocks, we must arrange the even numbers in each block in decreasing order so that there are a total $n!$ ways to partition the even numbers into the blocks. Once we have arranged the even numbers into blocks, it is easy to see that we can arrange blocks $0, \ldots, n-1$ in any order an still get a permutation σ with $\overleftarrow{des}_E(\sigma) = n$. It thus follows that there are $(n!)^2$ such permutations. Thus $R_{n,2n} = (n!)^2$. □

Theorem 3. *We have* $R_{k,2n} = \binom{n}{k}^2 (n!)^2$.

Proof. It is easy to see from Theorem 1 that we have two following recursions for the coefficients $R_{k,n}$.

$$R_{k,2n+1} = (k+1)R_{k+1,2n} + (2n+1-k)R_{k,2n} \tag{1}$$

$$R_{k,2n+2} = (k+1)R_{k,2n+1} + (2n+2-k)R_{k-1,2n+1} \tag{2}$$

Iterating the recursions (1) and (2), we see that

$$\begin{aligned} R_{k,2n+2} &= (k+1)((k+1)R_{k+1,2n} + (2n+1-k)R_{k,2n}) \\ &\quad +(2n+2-k)((k)R_{k,2n} + (2n+2-k)R_{k-1,2n}) \\ &= (k+1)^2 R_{k+1,2n} + (2k(2n+1-k)+2n+1)R_{k,2n} + (2n+2-k)^2 R_{k-1,2n} \end{aligned}$$

Thus if we solve for $R_{k+1,2n}$, we get that

$$R_{k+1,2n} = \frac{R_{k,2n+2} - (2k(2n+1-k)+2n+1)R_{k,2n} - (2n+2-k)^2 R_{k-1,2n}}{(k+1)^2}. \tag{3}$$

Since we have the initial conditions that $R_{-1,2n} = 0$ and $R_{0,2n} = (n!)^2$ for all $n \geq 1$, it is a routine verification using one's favorite computer algebra system that $R_{k,2n} = \binom{n}{k}^2 (n!)^2$ is the unique solution to (3). □

It follows that we can also get a simple formula for $R_{k,2n+1}$ for all n and k. That is, it immediately follows from (1) that

Corollary 2. *One has*

$$R_{k,2n+1} = (k+1)\binom{n}{k+1}^2 (n!)^2 + (2n+1-k)\binom{n}{k}^2 (n!)^2 = \frac{1}{k+1}\binom{n}{k}^2 ((n+1)!)^2.$$

4 Ending with an even number: properties of $P_n(x,z)$

Let Θ_{2n} be the operator that sends z^0x^k to $(n+k+1)z^0x^k + (n-k)z^0x^{k+1}$ and sends z^1x^k to $(n+k+1)z^1x^k + z^0x^{k+1} + (n-k-1)z^1x^{k+1}$. Let Ω_{2n+1} be the operator that sends z^0x^k to $(n+k+1)z^0x^k + z^1x^k + (n-k)z^0x^{k+1}$ and sends z^1x^k to $(n+k+2)z^1x^k + (n-k)z^1x^{k+1}$. Then we have the following.

Theorem 4. *The polynomials $\{P_n(x,z)\}_{n\geq 1}$ satisfy the following recursions.*

1. *$P_1(x,z) = 1$ and $P_2(x,z) = 1+z$.*
2. *For all $n \geq 1$, $P_{2n+1}(x,z) = \Theta_{2n}(P_{2n}(x,z))$.*
3. *For all $n \geq 1$, $P_{2n+2}(x,z) = \Omega_{2n+1}(P_{2n+1}(x,z))$*

Proof. Part 1 is easy to verify by direct computation.

For part 2, suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{S}_{2n}$, $\overrightarrow{des}_E(\sigma) = k$ and $\sigma_1 \notin E$. It is then easy to see that if we insert $2n+1$ in position i where $i \in \overrightarrow{Des}_E(\sigma)$, then $\overrightarrow{des}_E(\sigma^{(i)}) = k$. Similarly, if we insert $2n+1$ in position i where either $i = 2n$ or σ_{i+1} is odd, then $\overrightarrow{des}_E(\sigma^{(i)}) = k$. However, if we insert $2n+1$ in position i where $i \notin \overleftarrow{Des}_E(\sigma)$ and σ_{i+1} is even, then $\overrightarrow{des}_E(\sigma^{(i)}) = k+1$. In every case, $\sigma^{(i)}$ will start with an odd number so that $\{\sigma^{(i)} : i = 0, \ldots, 2n\}$ gives a contribution of $(1+k+n)z^0x^k + (n-k)z^0x^{k+1}$ to $P_{2n+1}(x,z)$.

Similarly, suppose that $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{S}_{2n}$ and $\overrightarrow{des}_E(\sigma) = k$ and $\sigma_1 \in E$. It is then easy to see that if we insert $2n+1$ in position i where $i \in \overrightarrow{Des}_E(\sigma)$, then $\overrightarrow{des}_E(\sigma^{(i)}) = k$. Similarly, if we insert $2n+1$ in position $i > 0$ where either $i = 2n$ or σ_{i+1} is odd, then $\overrightarrow{des}_E(\sigma^{(i)}) = k$. If we insert $2n+1$ in position i where $i > 0$ and $i \notin \overleftarrow{Des}_E(\sigma)$ and σ_{i+1} is even, then $\overrightarrow{des}_E(\sigma^{(i)}) = k+1$. In all of these cases, $\sigma^{(i)}$ will start with an even number. However, if we insert $2n+1$ in position 0, then $\overrightarrow{des}_E(\sigma^{(i)}) = k+1$ but $\sigma^{(0)}$ will start with an odd number. Thus in this case, $\{\sigma^{(i)} : i = 0, \ldots, 2n\}$ gives a contribution of $(1+k+n)z^1x^k + (n-k-1)z^1x^{k+1} + z^0x^{k+1}$ to $P_{2n+1}(x,z)$.

For part 3, suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1} \in \mathcal{S}_{2n+1}$ and $\overrightarrow{des}_E(\sigma) = k$ and $\sigma_1 \notin E$. It is then easy to see that if we insert $2n+2$ in position i where $i \in \overrightarrow{Des}_E(\sigma)$, then $\overrightarrow{des}_E(\sigma^{(i)}) = k$. Similarly, if we insert $2n+2$ in position i where either $i = 2n+1$ or σ_{i+1} is odd and $i > 0$, then $\overrightarrow{des}_E(\sigma^{(i)}) = k$. If we insert $2n+2$ in position i where $i \notin \overleftarrow{Des}_E(\sigma)$ and σ_{i+1} is even, then $\overrightarrow{des}_E(\sigma^{(i)}) = k+1$. In all such cases, $\sigma^{(i)}$ will start with an odd number. However if we insert $2n+2$ in position 0, then $\overrightarrow{des}_E(\sigma^{(0)}) = k$ but $\sigma^{(0)}$ will start with an even number. Thus in this case, $\{\sigma^{(i)} : i = 0, \ldots, 2n+1\}$ gives a contribution of $(1+k+n)z^0x^k + zx^k + (n-k)z^0x^{k+1}$ to $P_{2n+2}(x,z)$.

Similarly, suppose that $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1} \in \mathcal{S}_{2n+1}$ and $\overrightarrow{des}_E(\sigma) = k$ and $\sigma_1 \in E$. It is then easy to see that if we insert $2n+2$ in position i where $i \in \overrightarrow{Des}_E(\sigma)$, then $\overrightarrow{des}_E(\sigma^{(i)}) = k$. Also, if we insert $2n+2$ in position i where either $i = 2n+1$ or σ_{i+1} is odd, then $\overrightarrow{des}_E(\sigma^{(i)}) = k$. If we insert $2n+2$ in position i where $i \notin \overleftarrow{Des}_E(\sigma)$ and σ_{i+1} is even, then $\overrightarrow{des}_E(\sigma^{(i)}) = k+1$. In all of these cases, $\sigma^{(i)}$ will start with an even number. Thus in this case, $\{\sigma^{(i)} : i = 0, \ldots, 2n\}$ gives a contribution of $(1+k+n+1)z^1x^k + (n-k)z^1x^{k+1}$ to $P_{2n+2}(x,z)$. $\square$

We can express Theorem 4 in terms of differential operators:

Corollary 3. *The polynomials $\{P_n(x,z)\}_{n\geq 1}$ are given by the following*

1. $P_1(x,z) = 1$, $P_2(x,z) = 1+z$, *and for all* $n \geq 1$,
2. $P_{2n+1}(x,z) = x(1-x)\frac{\partial}{\partial x}P_{2n}(x,z) + x(1-z)\frac{\partial}{\partial z}P_{2n}(x,z) + (1+n(1+x))P_{2n}(x,z)$ *and*
3. $P_{2n+2}(x,z) = x(1-x)\frac{\partial}{\partial x}P_{2n+1}(x,z) + z(1-z)\frac{\partial}{\partial z}P_{2n+1}(x,z) + (1+z+n(1+x))P_{2n+1}(x,z)$.

This given, we can easily compute the first few polynomials $P_n(x,z)$.

$P_1(x,z) = 1$

$P_2(x,z) = 1 + z$

$P_3(x,z) = 2 + 2z + 2x$

$P_4(x,z) = 4 + 8z + 8x + 4xz$

$P_5(x,z) = 12 + 24z + 48x + 24xz + 12x^2$

$P_6(x,z) = 36 + 108z + 216x + 216xz + 108x^2 + 36x^2z$

$P_7(x,z) = 144 + 432z + 1296x + 1296xz + 1296x^2 + 432x^2z + 144x^3$

$P_8(x,z) = 576 + 2304z + 6912x + 10368xz + 10368x^2 + 6192zx^2 + 2304x^3 + 576zx^3$

Theorem 5. *We have*

1. $P_{0,0,2n} = (n!)^2$.
2. $P_{1,0,2n} = (n!)(n+1)! - (n!)^2 = n(n!)^2$.
3. $P_{0,0,2n+1} = (n!)(n+1)!$.
4. $P_{1,0,2n+1} = ((n+1)!)^2 - (n!)(n+1)! = n\,(n!(n+1)!)$.

Proof. It is easy to see that the theorem holds for $n = 1$.

Now suppose that $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ is such that $\overrightarrow{des}_E(\sigma) = 0$. Then we can factor any such permutation into blocks by reading the permutation from right to left and cutting after each odd number. For example if $\sigma = 1\ 2\ 4\ 5\ 6\ 9\ 7\ 3\ 8\ 10$, then the blocks of σ would be 1 2 4, 5 6, 3, 9, 7, and 3 8 10. Since $\overrightarrow{des}_E(\sigma) = 0$, there may be a block of even numbers at the start which contains only even numbers that are arranged in increasing order. We call this final block the 0-th block. Every other block, when read from left to right, must start with an odd number $2k-1$ which can be followed by any subset of even numbers which are greater than $2k-1$ arranged in increasing order. We call such a block the k-th block. It is then easy to see that there are $(n+1)!$ ways to put the even numbers $2, 4, \ldots, 2n$ into the blocks. That is, $2n$ can go in any of the blocks $0, 1, \ldots, n$; $2(n-1)$ can go in any of the blocks $0, \ldots, n-1$, etc. More generally, $2(n-k)$ can go in any of blocks $0, \ldots, n-k$. Once we have arranged the even numbers into blocks, it is easy to see that we can arrange blocks $1, \ldots, n$ in any order and still get a permutation σ with $\overrightarrow{des}_E(\sigma) = 0$. It thus follows that there are $n!(n+1)!$ such permutations. Thus $P_{0,0,2n} + P_{1,0,2n} = n!(n+1)!$.

Now if we consider only permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n}$ such that $\overrightarrow{des}_E(\sigma) = 0$ and σ_1 is odd, then we cannot put any even numbers in the 0-th block and hence there are only $n!$ ways to put the even numbers into blocks. That is, $2n$ can go in any of the blocks $1, \ldots, n$; $2(n-1)$ can go in any of the blocks $1, \ldots, n-1$, etc. It follows that $P_{0,0,2n} = (n!)^2$ and that $P_{1,0,2n} = n!(n+1)! - (n!)^2 = n(n!)^2$.

Now suppose that $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1}$ is such that $\overrightarrow{des}_E(\sigma) = 0$. Then again we can factor any such permutation into blocks by reading the permutation from right to left and cutting after each odd number. For example if $\sigma = 11\ 1\ 2\ 4\ 5\ 6\ 9\ 7\ 3\ 8\ 10$, then the blocks of σ would be 11, 1 2 4, 5 6, 3, 9, 7, and 3 8 10. Since $\overrightarrow{des}_E(\sigma) = 0$, there may be a block of even numbers at the start which contains only even numbers that are arranged in increasing order. We call this final block the 0-th block. Every other block, when read from left to right, must start with an odd number $2k-1$ which can be followed by any subset of even numbers which are greater than $2k-1$ arranged in increasing order. We call such a block the k-th block. Notice that this forces the $(n+1)$-st block, i.e., the block which ends with $2n+1$ to have no even numbers in it. It is then easy to see that there are $(n+1)!$ ways to put the even numbers $2, 4, \ldots, 2n$ into the blocks. That is, $2n$ can go in any of the blocks $0, 1, \ldots, n$; $2(n-1)$ can go in any of the blocks $0, \ldots, n-1$, etc. More generally, $2(n-k)$ can go in any blocks $0, \ldots, n-k$. Once we have arranged the even numbers into blocks, it is easy to see that we can arrange blocks $1, \ldots, n+1$ in any order and still get a permutation σ with $\overrightarrow{des}_E(\sigma) = 0$. It thus follows that there are $((n+1)!)^2$ such permutations. Thus $P_{0,0,2n+1} + P_{1,0,2n+1} = ((n+1)!)^2$.

Now if we consider only permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1}$ such that $\overrightarrow{des}_E(\sigma) = 0$ and σ_1 is odd, then we cannot put any even numbers in the 0-th block and hence there are only $n!$ ways to put the even numbers into blocks. That is, $2n$ can go in any of the blocks $1, \ldots, n$; $2(n-1)$ can go in any of the blocks $1, \ldots, n-1$, etc. It follows that $P_{0,0,2n} = n!(n+1)!$ and that $P_{1,0,2n} = ((n+1)!)^2 - n!(n+1)! = n(n!(n+1)!)$. $\square$

Theorem 6. *For all $0 \leq k \leq n$,*

$$\begin{aligned} P_{0,k,2n+1} &= P_{0,n-k,2n+1} \text{ and} \\ P_{1,k,2n+1} &= P_{1,n-k-1,2n+1} \end{aligned}$$

Proof. Now suppose that σ_1 is odd. Then $2n+2-\sigma_1$ is odd and, for any even number $2m$, if $\sigma_i = 2m$, then $i > 1$ and $i-1 \in \overrightarrow{Des}_E(\sigma) \iff i-1 \notin \overrightarrow{Des}_E(\sigma^c)$. It follows that $\overrightarrow{des}_E(\sigma) = k \iff \overrightarrow{des}_E(\sigma^c) = n-k$. Hence the complementation map shows $P_{0,k,2n+1} = P_{0,n-k,2n+1}$.

Similarly, suppose that σ_1 is even. Then $2n+2-\sigma_1$ is even. Moreover, there are $n-1$ even numbers $2m$ such that there exists an $i > 1$ such that $\sigma_i = 2m$ and, for any such even number $2m$, $i-1 \in \overrightarrow{Des}_E(\sigma) \iff i-1 \notin \overrightarrow{Des}_E(\sigma^c)$. It follows that $\overrightarrow{des}_E(\sigma) = k \iff \overrightarrow{des}_E(\sigma^c) = n-1-k$. Hence the complementation map shows $P_{1,k,2n+1} = P_{1,n-k-1,2n+1}$. $\square$

Here is a result about the relationships between the polynomials $R_n(x)$ and the polynomials $P_n(x,z)$

Theorem 7. *For all k and n, $R_{k,2n+1} = P_{0,k,2n+1} + P_{1,k,2n+1}$ so that $R_{2n+1}(x) = P_{2n+1}(x,1)$.*

Proof. This result follows by sending σ to the reverse of σ^c. That is, if $\sigma_i > \sigma_{i+1}$ and $\sigma_i \in E$, then $\sigma_i^c < \sigma_{i+1}^c$ and $\sigma_i^c \in E$ and, hence, in the reverse of σ^c, σ_i^c will be part of descent that ends in an even number. $\square$

Theorem 8. *For all* $0 \le k \le n$,

$$P_{1,k,2n} = \binom{n-1}{k}\binom{n}{k+1}(n!)^2$$

and

$$P_{0,k,2n} = \binom{n-1}{k}\binom{n}{k}(n!)^2$$

Proof. It follows from Theorem 4 that we have the following recursions for the coefficients of $P_{2n+1}(x,z)$ and $P_{2n+2}(x,z)$.

$$P_{0,k,2n+1} = (n+k+1)P_{0,k,2n} + P_{1,k-1,2n} + (n-k+1)P_{0,k-1,2n}, \tag{4}$$

$$P_{1,k,2n+1} = (n+k+1)P_{1,k,2n} + (n-k)P_{1,k-1,2n}, \tag{5}$$

$$P_{0,k,2n+2} = (n+k+1)P_{0,k,2n+1} + (n-k+1)P_{0,k-1,2n+1}, \tag{6}$$

and

$$P_{1,k,2n+2} = (n+k+2)P_{1,k,2n+1} + P_{0,k,2n+1} + (n-k+1)P_{1,k-1,2n+1}.$$

Thus it follows that

$$\begin{aligned} P_{0,k,2n+2} &= (n+k+1)\left((n+k+1)P_{0,k,2n} + P_{1,k-1,2n} + (n-k+1)P_{0,k-1,2n}\right) \\ &\quad +(n-k+1)\left((n+k)P_{0,k-1,2n} + P_{1,k-2,2n} + (n-k+2)P_{0,k-2,2n}\right) \\ &= (n+k+1)^2P_{0,k,2n} + (2n^2+3n-2k^2+k+1)P_{0,k-1,2n} \\ &\quad +(n-k+1)(n-k+2)P_{0,k-2,2n} \\ &\quad +(n+k+1)P_{1,k-1,2n} + (n-k+1)P_{1,k-2,2n}. \end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned} P_{1,k,2n+2} &= (n+k+2)\left((n+k+1)P_{1,k,2n} + (n-k)P_{1,k-1,2n}\right) \\ &\quad +(n+k+1)P_{0,k,2n} + P_{1,k-1,2n} + (n-k+1)P_{0,k-1,2n} \\ &\quad +(n-k+1)\left((n+k)P_{1,k-1,2n} + (n-k+1)P_{1,k-2,2n}\right) \\ &= (n+k+1)(n+k+2)P_{1,k,2n} + (2n^2+3n-2k^2-k+1)P_{1,k-1,2n} \\ &\quad +(n-k+1)^2P_{1,k-2,2n} \\ &\quad +(n+k+1)P_{0,k,2n} + (n-k+1)P_{0,k-1,2n}. \end{aligned} \tag{8}$$

We note that our proof of Theorem 5 shows that our formulas hold for all n when $k=0$. It is also easy to check that our formulas hold when $n=1$ for all k. Next we consider the case when $k=1$. In this case $P_{0,1-2,2n} = P_{1,1-2,2n} = 0$ for all n by definition so that the recursion (7) reduces to

$$P_{0,1,2n+2} = (n+2)^2P_{0,1,2n} + (2n^2+3n)P_{0,0,2n} + (n+2)P_{1,0,2n}. \tag{9}$$

Given that $P_{0,0,2n} = (n!)^2$ and $P_{1,0,2n} = n(n!)^2$ by Theorem 5, assuming by induction that $P_{0,1,2n} = (n-1)(n)(n!)^2$, and using (9), we obtain that

$$\begin{aligned} P_{0,1,2n+2} &= (n+2)^2(n-1)n(n!)^2 + (2n^2+3n)(n!)^2 + (n+2)n(n!)^2 \\ &= n(n+1)((n+1)!)^2. \end{aligned}$$

Thus our formula for $P_{0,1,2n}$ holds by induction.

Similarly the recursion (8) reduces to

$$P_{1,1,2n+2} = (n+3)(n+2)P_{1,1,2n} + (2n^2+3n-2)P_{1,0,2n} + (n+2)P_{0,1,2n} + nP_{0,0,2n}. \quad (10)$$

Note that $P_{0,0,2n} = (n!)^2$ and $P_{1,0,2n} = n(n!)^2$ by Theorem 5 and we just proved that $P_{0,1,2n} = (n-1)n(n!)^2$. Thus if we assume by induction that $P_{1,1,2n} = (n-1)\binom{n}{2}(n!)^2$ then using (10), we obtain that

$$\begin{aligned} P_{1,1,2n+2} &= (n+3)(n+2)(n-1)\binom{n}{2}(n!)^2 + (2n^2+3n-2)n(n!)^2 \\ &\quad +(n+2)(n-1)n(n!)^2 + n(n!)^2 = n\binom{n+1}{2}((n+1)!)^2. \end{aligned}$$

Thus our formula for $P_{1,1,2n}$ holds by induction.

We now are in a position to prove the general cases of our formulas. That is, we shall prove our formulas hold for all n by induction on k. That is, assume our formulas hold for all n and for all $j < k$ and that they also hold for $2n$ and k. Then, using (7),

$$\begin{aligned} P_{0,k,2n+2} &= (n+k+1)^2\binom{n-1}{k}\binom{n}{k}(n!)^2 + (2n^2+3n-2k^2+k+1)\binom{n-1}{k-1}\binom{n}{k-1}(n!)^2 \\ &\quad +(n-k+1)(n-k+2)\binom{n-1}{k-2}\binom{n}{k-2}(n!)^2 \\ &\quad +(n+k+1)\binom{n-1}{k-1}\binom{n}{k}(n!)^2 + (n-k+1)\binom{n-1}{k-2}\binom{n}{k-1}(n!)^2. \end{aligned} \quad (11)$$

Thus we have to show that

$$\binom{n}{k}\binom{n+1}{k}((n+1)!)^2 = \text{(the RHS of (11))}. \quad (12)$$

We can clearly divide both sides of (12) by $(n-1)!n!(n!)^2$ to obtain

$$\begin{aligned}\frac{n(n+1)^3}{k!(n-k)!k!(n-k+1)!} &= \frac{(n+k+1)^2}{k!(n-k-1)!k!(n-k)!} \\ &+\frac{2n^2+3n-2k^2+k+1}{(k-1)!(n-k)!(k-1)!(n-k+1)!} \\ &+\frac{(n-k+2)(n-k+1)}{(k-2)!(n-k+1)!(k-2)!(n-k+2)!} \\ &+\frac{(n+k+1)}{(k-1)!(n-k)!(k)!(n-k)!} \\ &+\frac{n-k+1}{(k-2)!(n-k+1)!(k-1)!(n-k)!}.\end{aligned}$$

We multiply both sides of (13) by $k!(n-k)!k!(n-k+1)!$ to get a new identity which is easy to check.

Given that we have proved our formula for $P_{0,k,2n}$, we can prove our formula for $P_{1,k,2n}$ in a similar manner. That is, we shall prove our formula for $P_{1,k,2n}$ holds for all n by induction on k. Assume our formula hold for all n and for all $j < k$ and that it also holds for $2n$ and k. Then, using 8,

$$\begin{aligned}P_{1,k,2n+2} &= (n+k+1)(n+k+2)\binom{n-1}{k}\binom{n}{k+1}(n!)^2 \\ &+(2n^2+3n-2k^2-k+1)\binom{n-1}{k-1}\binom{n}{k}(n!)^2 \\ &+(n-k+1)^2\binom{n-1}{k-2}\binom{n}{k-1}(n!)^2 \\ &+(n+k+1)\binom{n-1}{k}\binom{n}{k}(n!)^2 \\ &+(n-k+1)\binom{n-1}{k-1}\binom{n}{k-1}(n!)^2 \qquad (13)\end{aligned}$$

Then we must show that

$$\binom{n}{k}\binom{n+1}{k+1}((n+1)!)^2 = (\text{the RHS of (13)}). \qquad (14)$$

It is easy to see that we can factor out $(n-1)!n!(n!)^2$ from both side of (14) to obtain that

$$\begin{aligned}
\frac{n(n+1)^3}{k!(n-k)!(k+1)!(n-k)!} &= \frac{(n+k+2)(n+k+1)}{k!(n-k-1)!(k+1)!(n-k-1)!} \\
&\quad + \frac{2n^2+3n-2k^2-k+1}{(k-1)!(n-k)!(k)!(n-k)!} \\
&\quad + \frac{(n-k+1)^2}{(k-2)!(n-k+1)!(k-1)!(n-k+1)!} \\
&\quad + \frac{(n+k+1)}{(k)!(n-k-1)!(k-1)!(n-k)!} \\
&\quad + \frac{(n-k+1)}{(k-1)!(n-k)!(k-1)!(n-k+1)!}. \qquad (15)
\end{aligned}$$

If we cancel the terms $(n-k+1)$ from both the numerator and denominator of the third and fifth terms on the RHS of (15), then it is easy to see that we can multiple both sides of (15) by $k!(n-k)!(k+1)!(n-k)!$ to obtain an identity that is easy to check. $\square$

Having found formulas for $P_{j,k,n}$ for the even values of n, we can easily use the recursions (4) and (5) to find formulas for $P_{j,k,n}$ for the odd values of n.

Theorem 9. *For all* $0 \leq k \leq n$*,*

$$P_{0,k,2n+1} = (k+1)\binom{n}{k}\binom{n+1}{k+1}(n!)^2 = (n+1)\binom{n}{k}^2(n!)^2 \qquad (16)$$

and

$$P_{1,k,2n+1} = \frac{(n+1)(n-k)}{k+1}\binom{n}{k}^2(n!)^2.$$

Proof. Using the recursion (4) and Theorem 8, we have

$$\begin{aligned}
P_{0,k,2n+1} &= (n+k+1)\binom{n-1}{k}\binom{n}{k}(n!)^2 \\
&\quad + \binom{n-1}{k-1}\binom{n}{k}(n!)^2 \\
&\quad + (n-k+1)\binom{n-1}{k-1}\binom{n}{k-1}(n!)^2 \\
&= \binom{n}{k}(n!)^2\left(n\binom{n-1}{k} + (k+1)\binom{n}{k}\right)
\end{aligned}$$

Here we have used the identities that $(n-k+1)\binom{n}{k-1} = k\binom{n}{k}$ and $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$. It is then easy to verify that

$$n\binom{n-1}{k} + (k+1)\binom{n}{k} = (n+1)\binom{n}{k}$$

and we get desired.

By Theorem 7 and Corollary 2, we have that

$$R_{k,2n+1} = P_{0,k,2n+1} + P_{1,k,2n+1} = \frac{1}{k+1}\binom{n}{k}^2((n+1)!)^2.$$

Thus, using (16), we get that

$$P_{1,k,2n+1} = \frac{1}{k+1}\binom{n}{k}^2((n+1)!)^2 - (n+1)\binom{n}{k}^2(n!)^2$$

which leads to the result after a simplification.

□

As a corollary to Theorem 8, one has $P_{0,k,2n} = P_{1,n-1-k,2n}$, and, using Theorem 3, we get that

$$R_{k,2n} = P_{0,k,2n} + P_{1,k-1,2n}.$$

5 Ending with an odd number: properties of $Q_n(x,z)$

Let Φ_{2n} be the operator that sends z^0x^k to $z^1x^k + (n+k)z^0x^k + (n-k)z^0x^{k+1}$ and sends z^1x^k to $(n+k+1)z^1x^k + (n-k)z^1x^{k+1}$. Let Ψ_{2n+1} be the operator that sends z^0x^k to $(n+k+1)z^0x^k + (n-k+1)z^0x^{k+1}$ and sends z^1x^k to $(n+k+1)z^1x^k + z^0x^{k+1} + (n-k)z^1x^{k+1}$. Then we have the following.

Theorem 10. *The polynomials $\{Q_n(x,z)\}_{n\geq 1}$ satisfy the following recursions.*

1. *$Q_1(x,z) = z$ and $Q_2(x,z) = z + x$.*

2. *For all $n \geq 1$, $Q_{2n+1}(x,z) = \Phi_{2n}(Q_{2n}(x,z))$.*

3. *For all $n \geq 1$, $Q_{2n+2}(x,z) = \Psi_{2n+1}(Q_{2n+1}(x,z))$*

Proof. Part 1 is easy to verify by direct computation.

For part 2, suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{S}_{2n}$, $\overrightarrow{des}_O(\sigma) = k$ and $\sigma_1 \notin O$. It is then easy to see that if we insert $2n+1$ in position i where $i \in \overrightarrow{Des}_O(\sigma)$, then $\overrightarrow{des}_O(\sigma^{(i)}) = k$. Similarly, if we insert $2n+1$ in position i where either $i = 2n$ or σ_{i+1} is even, then $\overrightarrow{des}_O(\sigma^{(i)}) = k$. Notice, that inserting 2n+1 in position 0 produces $\sigma^{(0)}$ starting from an odd number giving a contribution of zx^k rather than x^k to $Q_{2n+1}(x,z)$. However, if we insert $2n+1$ in position i in front of remaining odd numbers then $\overrightarrow{des}_O(\sigma^{(i)}) = k+1$. In every case but one, $\sigma^{(i)}$ starts with an even number so that $\{\sigma^{(i)} : i = 0,\ldots,2n\}$ gives a contribution of $z^1x^k + (k+1+(n-1))z^0x^k + (n-k)z^0x^{k+1}$.

Similarly, suppose that $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{S}_{2n}$, $\overrightarrow{des}_O(\sigma) = k$ and $\sigma_1 \in O$. It is then easy to see that if we insert $2n+1$ in position i where $i \in \overrightarrow{Des}_O(\sigma)$, then $\overrightarrow{des}_O(\sigma^{(i)}) = k$.

Similarly, if we insert $2n+1$ in position i where either $i = 2n$ or σ_{i+1} is even, then $\overrightarrow{des}_O(\sigma^{(i)}) = k$. If we insert $2n+1$ in position i where $i \notin \overleftarrow{Des}_O(\sigma)$ and σ_{i+1} is odd, then $\overrightarrow{des}_O(\sigma^{(i)}) = k+1$. In all of these cases, $\sigma^{(i)}$ will start with an odd number. Thus in this case, $\{\sigma^{(i)} : i = 0,\ldots,2n\}$ gives a contribution of $(n+k+1)z^1x^k + (n-k)z^1x^{k+1}$ to $Q_{2n+1}(x,z)$.

For part 3, suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1} \in \mathcal{S}_{2n+1}$, $\overrightarrow{des}_O(\sigma) = k$ and $\sigma_1 \notin O$. It is then easy to see that if we insert $2n+2$ in position i where $i \in \overrightarrow{Des}_O(\sigma)$, then $\overrightarrow{des}_O(\sigma^{(i)}) = k$. Similarly, if we insert $2n+2$ in position i where either $i = 2n+1$ or σ_{i+1} is even, then $\overrightarrow{des}_O(\sigma^{(i)}) = k$. If we insert $2n+2$ in position i where $i \notin \overleftarrow{Des}_O(\sigma)$ and σ_{i+1} is odd, then $\overrightarrow{des}_O(\sigma^{(i)}) = k+1$. In all such cases, $\sigma^{(i)}$ will start with an even number. Thus in this case, $\{\sigma^{(i)} : i = 0,\ldots,2n+1\}$ gives a contribution of $(1+k+n)z^0x^k + (n-k+1)z^0x^{k+1}$ to $Q_{2n+2}(x,z)$.

Similarly, suppose that $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1} \in \mathcal{S}_{2n+1}$, $\overrightarrow{des}_O(\sigma) = k$ and $\sigma_1 \in O$. It is then easy to see that if we insert $2n+2$ in position i where $i \in \overrightarrow{Des}_O(\sigma)$, then $\overrightarrow{des}_O(\sigma^{(i)}) = k$. Also, if we insert $2n+2$ in position i where either $i = 2n+1$ or σ_{i+1} is even, then $\overrightarrow{des}_O(\sigma^{(i)}) = k$. If we insert $2n+2$ in position $i > 0$ where $i \notin \overleftarrow{Des}_O(\sigma)$ and σ_{i+1} is odd, then $\overrightarrow{des}_O(\sigma^{(i)}) = k+1$. In all of these cases, $\sigma^{(i)}$ will start with an odd number. However, $\sigma^{(0)}$ starts with an even number and $\overrightarrow{des}_O(\sigma^{(0)}) = k+1$. Thus in this case, $\{\sigma^{(i)} : i = 0,\ldots,2n\}$ gives a contribution of $(1+k+n)z^1x^k + x^{k+1} + (n-k)z^1x^{k+1}$ to $Q_{2n+2}(x,z)$. $\square$

We can express Theorem 10 in terms of differential operators:

Corollary 4. *The polynomials $\{Q_n(x,z)\}_{n\geq 1}$ are given by the following*

1. $Q_1(x,z) = z$, $Q_2(x,z) = z + x$, *and for all* $n \geq 1$,
2. $Q_{2n+1}(x,z) = x(1-x)\frac{\partial}{\partial x}Q_{2n}(x,z) + z(1-z)\frac{\partial}{\partial z}Q_{2n}(x,z) + (z + n(1+x))Q_{2n}(x,z)$
3. $Q_{2n+2}(x,z) = x(1-x)\frac{\partial}{\partial x}Q_{2n+1}(x,z) + x(1-z)\frac{\partial}{\partial z}Q_{2n+1}(x,z) + (1+n)(1+x)Q_{2n+1}(x,z)$.

This given, we can easily compute the first few polynomials $Q_n(x,z)$.

$$Q_1(x,z) = z$$

$$Q_2(x,z) = z + x$$

$$Q_3(x,z) = 2z + 2x + 2xz$$

$$Q_4(x,z) = 4z + 8x + 8zx + 4x^2$$

$$Q_5(x,z) = 12z + 24x + 48xz + 24x^2 + 12x^2z$$

$$Q_6(x,z) = 36z + 108x + 216xz + 216x^2 + 108x^2z + 36x^3$$

$$Q_7(x,z) = 144z + 432x + 1296xz + 1296x^2 + 1296x^2z + 432x^3 + 144x^3z$$

$$Q_8(x,z) = 576z + 2304x + 6912xz + 10368x^2 + 10368x^2z + 6192x^3 + 2304x^3z + 576x^4$$

Forms of the polynomials $P_n(x,z)$ and $Q_n(x,z)$ suggest the following result.

Theorem 11. *For $n \geq 1$, $Q_n(x,z) = \Xi(P_n(x,z))$, where Ξ is the operator that sends z^0x^k to z^1x^k and z^1x^k to x^{k+1}. In other words, the number of n-permutations σ beginning with an odd number and having $\overrightarrow{des}_E(\sigma) = k$ is equal to that of n-permutations π beginning with an odd number and having $\overrightarrow{des}_O(\pi) = k$; also, the number of n-permutations σ beginning with an even number and having $\overrightarrow{des}_E(\sigma) = k$ is equal to that of n-permutations π beginning with an even number and having $\overrightarrow{des}_O(\pi) = k+1$.*

Proof. We prove the result by induction on n. The statement is true for $n = 1, 2$, which can be seen directly from the polynomials. Suppose the statement is true for $2n$, that is, $Q_{2n}(x,z) = \Xi(P_{2n}(x,z))$. Now $Q_{2n+1}(x,z) = \Phi_{2n}(Q_{2n}(x,z)) = \Phi_{2n}(\Xi(P_{2n}(x,z)))$ and we want to show this to be equal to $\Xi(P_{2n+1}(x,z)) = \Xi(\Theta_{2n}(P_{2n}(x,z)))$. In other words, we want to prove that the operator $\Phi_{2n}(\Xi(\cdot))$ is identical to the operator $\Xi(\Theta_{2n}(\cdot))$ which can be checked directly by finding the images of z^0x^k and z^1x^k.

Suppose now the statement is true for $2n+1$. One can use considerations as above involving proving that the operator $\Psi_{2n+1}(\Xi(\cdot))$ is identical to the operator $\Xi(\Omega_{2n+1}(\cdot))$, to show that the statement is also true for $2n+2$. □

The following corollary to Theorem 11 is easy to see.

Corollary 5. *For all $0 \leq k \leq n$, $P_{0,k,n} = Q_{1,k,n}$ and $P_{1,k,n} = Q_{0,k+1,n}$, and thus*

$$\begin{aligned}
Q_{0,k,2n} &= \tbinom{n-1}{k-1}\tbinom{n}{k}(n!)^2,\\
Q_{0,k,2n+1} &= \tfrac{(n+1)(n-k+1)}{k}\tbinom{n}{k-1}^2(n!)^2,\\
Q_{1,k,2n} &= \tbinom{n-1}{k}\tbinom{n}{k}(n!)^2,\\
Q_{1,k,2n+1} &= \tbinom{n}{k}^2 n!(n+1)!.
\end{aligned}$$

6 Beginning with an odd number: properties of $M_n(x)$

Theorem 12. *For all $0 \leq k \leq n$,*

$$M_{k,2n} = \frac{n+1}{k+1}\binom{n-1}{k}\binom{n}{k}(n!)^2 = \binom{n-1}{k}\binom{n+1}{k+1}(n!)^2 \tag{17}$$

and

$$M_{k,2n+1} = \frac{1}{n-k+1}\binom{n}{k}^2((n+1)!)^2 = \binom{n}{k}\binom{n+1}{k}n!(n+1)!. \tag{18}$$

Proof. To prove the first result, note that $M_{k,2n} = P_{0,k,2n} + P_{1,k,2n}$ since there is a one-to-one correspondence between permutations counted by $M_{k,2n}$ and the $2n$-permutations σ with $\overrightarrow{des}_E(\sigma) = k$. Indeed, a bijection is given by taking the reverse and then the complement (the parity of the letters will be changed). Now we simply apply Theorem 8 and simplify $P_{0,k,2n} + P_{1,k,2n}$.

To prove the second result, note that $M_{k,2n+1} = Q_{0,k,2n+1} + Q_{1,k,2n+1}$ since there is a one-to-one correspondence between permutations counted by $M_{k,2n+1}$ and the $(2n+1)$-permutations σ with $\overrightarrow{des}_O(\sigma) = k$. Indeed, a bijection is given by taking the reverse and then the complement (the parity of the letters will be unchanged). Now we simply use Corollary 5 and simplify $Q_{0,k,2n+1} + Q_{1,k,2n+1}$. □

7 Connection to the Genocchi numbers

Probably the study of *Genocchi numbers* goes back to Euler. The Genocchi numbers can be defined by the following generating function.

$$\frac{2t}{e^t+1} = t + \sum_{n\geq 1}(-1)^n G_{2n}\frac{t^{2n}}{(2n)!}. \tag{19}$$

These numbers were studied intensively during the last three decades (see, e.g., [5] and references therein). Dumont [4] showed that the Genocchi number G_{2n} is the number of permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1}$ in $\mathcal{S}_{2n+1}$ such that

$$\begin{array}{ll} \sigma_i < \sigma_{i+1} & \text{if } \sigma_i \text{ is odd,} \\ \sigma_i > \sigma_{i+1} & \text{if } \sigma_i \text{ is even.} \end{array}$$

The first few Genocchi numbers are $1, 1, 3, 17, 155, 2073, \ldots$.

Study of distribution of descents is the same as study of distribution of consecutive occurrences of the *pattern* 21 *with no dashes* (see, e.g., [1] for terminology). Likewise, distribution of descents according to parity can be viewed as distribution of consecutive occurrences of certain patterns.

Let us fix some notations. We use e, o, or $*$ as superscripts for a pattern's letters to require that in an occurrence of the pattern in a permutation, the corresponding letters must be even, odd or either. For example, the permutation 25314 has two occurrences of the pattern 2^*1^o (they are 53 and 31, both of them are occurrences of the pattern 2^o1^o), one occurrence of the pattern 1^o2^e (namely, 14), no occurrences of the pattern 1^o2^o, and no occurrences of the pattern 2^e1^*.

Given this notation, we can state an alternative definition of the Genocchi numbers, which follows directly from the definition above:

Definition 1. *The Genocchi number G_{2n} is the number of permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1}$ in $\mathcal{S}_{2n+1}$ that avoid simultaneously the patterns 1^e2^* and 2^o1^*.*

Our terminology allows to define the Genocchi numbers on even permutations as well, due to the following result which we state as a conjecture (note that in the conjecture only descents according to parity are involved unlike Definition 1):

Conjecture 1. *The number of permutations in $\mathcal{S}_{2n}$ that avoid simultaneously the patterns 2^*1^e and 2^e1^* is given by the Genocchi number G_{2n}.*

As a consequence of Conjecture 1, one can get an alternative definition of the Genocchi numbers: For $n \geq 1$, the number of permutations in $\mathcal{S}_{2n-1}$ that avoid simultaneously the patterns 2^*1^e and 2^e1^* is given by G_{2n}. Indeed, we can use an observation that the number $2n$ must be at the end of a permutation avoiding 2^*1^e and 2^e1^*, and adjoining this number from the right to any "good" $(2n-1)$-permutation gives a "good" $(2n)$-permutation.

The following theorem provides one more alternative definition of the Genocchi numbers.

Theorem 13. *The number of permutations in $\mathcal{S}_{2n}$ that avoid simultaneously the patterns 1^e2^* and 2^o1^* is given by $2G_{2n}$.*

Proof. We note the following: the number $2n+1$ must be the rightmost number of a $(2n+1)$-permutation avoiding 1^e2^* and 2^o1^*. Clearly, if we remove this number from a "good" permutation, we get a "good" $(2n)$-permutation. The reverse is not true, since if a "good" $(2n)$-permutation ends with an even number, then adjoining $2n+1$ to the right of it gives an occurrence of 1^e2^*, whereas adjoining $2n+1$ to the right of an odd number does not lead to an occurrence of a prohibited pattern. Thus to prove the statement, we need to prove that among 1^e2^*- and 2^o1^*-avoiding $(2n)$-permutations half end with an even number, and half end with an odd number. This, however, is easy to see since a bijection between these objects is given by the complement. Indeed, the complement changes parity (in particular parity of the rightmost number), and also σ avoids 1^e2^* and 2^o1^* if and only if its complement σ^c, avoids 1^e2^* and 2^o1^*. □

To summarize the section we say that certain descents according to parity do not only provide alternative definitions for the Genocchi numbers but also generalize them in sense that instead of considering (multi-)avoidance of the descents, which gives the Genocchi numbers, one may consider, for example, their (join) distribution.

8 Bijective proofs related to the context

It is always nice to be able to provide a bijective proof for an identity. From the form of coefficients of some polynomials in this paper, one can see several relations between different groups of permutations. In this section we provide bijective solutions for five such relations.

For Subsections 8.4 and 8.5, recall that if σ is a permutation of $\mathcal{S}_n$ then $\sigma^{(i)}$ denotes the permutation of $\mathcal{S}_{n+1}$ that results by inserting $n+1$ in position i.

8.1 Bijective proof for the symmetry of $R_{2n}(x)$.

Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{S}_{2n}$ with $\overleftarrow{des}_E(\sigma) = k$, apply the complement to the permutation $\sigma' = \sigma(2n+1)$, that is, σ' is obtained from σ by adding a dummy number

$(2n+1)$ at the end. In the obtained permutation σ'^c, make a cyclic shift to the left to make the number $(2n+1)$ be the first one. Remove $(2n+1)$ to get a $2n$-permutation σ^* with $\overleftarrow{des}_E(\sigma^*) = n-k$. To reverse this procedure, adjoin $(2n+1)$ from the left to a given permutation σ^* with $\overleftarrow{des}_E(\sigma^*) = n-k$. Then make a cyclic shift to the right to make 1 be the rightmost number. Use the complement and remove $(2n+1)$ from the obtained permutation to get a permutation σ with $\overleftarrow{des}_E(\sigma) = k$.

The map described above and its reverse are clearly injective. We only need to justify that given k occurrences of descents in σ ("descents" in this subsection are those from $\overleftarrow{Des}_E(\pi)$ for some π), we get $(n-k)$ occurrences in σ^* (the reverse to this statement will follow using the same arguments). Notice that adding $(2n+1)$ at the end does not increase the number of descents. Since σ' ends with an odd number, σ'^c has $n-k$ descents. If $\sigma' = A1B(2n+1)$, where A and B are some factors, then $\sigma'^c = A^c(2n+1)B^c1$ and σ^* is $(2n+1)B^c1A^c$ without $(2n+1)$. The last thing to observe is that moving A^c to the end of σ'^c does not create a new descent since it cannot start with 1, also we do not lose any descents since none of them can end with $(2n+1)$. So, σ^* has $n-k$ descents.

8.2 Bijective proof for $R_{k,2n} = P_{0,k,2n} + P_{1,k-1,2n}$.

A similar solution as that in Subsection 8.1 works, with the main difference that we now keep track of whether this is an odd or even number to the left of 1 in $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{S}_{2n}$ with $\overleftarrow{des}_E(\sigma) = k$. We do not provide all the justifications in our explanation of the map, since they are similar to that in Subsection 8.1.

Suppose $\sigma' = \sigma(2n+1) = Ax1B(2n+1)$, where A and B are some factors and x is a number. Apply the complement *and reverse* to σ' to get $(\sigma')^{cr} = 1B^{cr}(2n+1)x^cA^{cr}$. Make a cyclic shift to the left in $(\sigma')^{cr}$ to make the number $(2n+1)$ be the first one and to get $\sigma^* = (2n+1)x^cA^{cr}1B^{cr}$. One can check that $\overleftarrow{des}_E(\sigma) = \overrightarrow{des}_E(\sigma^*) = k$. Also, parity of x is the same as parity of x^c. Now, if we remove $(2n+1)$ from σ^* and x^c is even, we loose one descent obtaining a permutation counted by $P_{1,k-1,2n}$; if we remove $(2n+1)$ from σ^* and x^c is odd, the number of descents in the obtained permutation is the same, k, and thus we get a permutation counted by $P_{0,k,2n}$. Note that if Ax is the empty word, that is, σ starts with 1, then this case is treated as the case "x is odd" since σ^* will start with $(2n+1)1$. Thus one may think of $2n+1$ as the (cyclic) predecessor of 1 in this case, that is, $x = 2n+1$.

The reverse to the map described is easy to see.

8.3 Bijective proof for $P_{0,k,2n} = P_{1,n-1-k,2n}$.

This identity states that the number of $2n$-permutations σ beginning with an odd number and having $\overrightarrow{des}_E(\sigma) = k$ is equal to that of $2n$-permutations π beginning with an even number and having $\overrightarrow{des}_E(\pi) = n-1-k$. Let $\mathcal{A}_{2n}(k)$ (resp. $\mathcal{A}_{2n}(n-1-k)$) denote the set of permutations of the first (resp. second) kind.

Suppose $\pi = x\pi' \in \mathcal{A}_{2n}(n-1-k)$ where x is an even number. Adjoin a dummy number $2n+1$ to π from the right to get the permutation $\pi_1 = (2n+1)x\pi'$ with $\overrightarrow{des}_E(\pi_1) = n-k$. Apply the compliment to get the $(2n+1)$-permutation $\pi_2 = \pi_1^c = 1x^c(\pi')^c$. Since parity of the numbers preserved after applying the complement, clearly $\overrightarrow{des}_E(\pi_2) = k$ and x^c is an even number. Now remove 1 from π_2 and decrease each number of π_2 by 1 to get the $2n$-permutation $\pi_3 = y\pi''$ where $y = x^c - 1$ is an odd number and $\overrightarrow{des}_O(\pi_3) = k$ (each descent ending with an even number becomes a descent ending with an odd number). Using the notation from Subsection 8.4, $\pi_3 \in \mathcal{B}_{2n}(k)$, which is the set of $2n$-permutations τ beginning with an odd number and having $\overrightarrow{des}_O(\tau) = k$.

Since all the steps made above are invertible, we have a bijection between $\mathcal{A}_{2n}(n-1-k)$ and $\mathcal{B}_{2n}(k)$. It now remains to apply the (recursive) bijection between $\mathcal{A}_{2n}(k)$ and $\mathcal{B}_{2n}(k)$ provided in Subsection 8.4, to get the desired bijection between $\mathcal{A}_{2n}(k)$ and $\mathcal{A}_{2n}(n-1-k)$.

8.4 Bijective (recursive) proof for $P_{0,k,n} = Q_{1,k,n}$.

This identity states that the number of n-permutations σ beginning with an odd number and having $\overrightarrow{des}_E(\sigma) = k$ is equal to that of n-permutations π beginning with an odd number and having $\overrightarrow{des}_O(\pi) = k$. Let $\mathcal{A}_n(k)$ (resp. $\mathcal{B}_n(k)$) denote the set of permutations of the first (resp. second) kind. Of course, $\cup_{k\geq 0}\mathcal{A}_n(k) = \cup_{k\geq 0}\mathcal{B}_n(k)$ is the set of all permutations in $\mathcal{S}_n$ that begin with an odd number.

For $\mathcal{S}_1$ we map $1 \in \mathcal{A}_1(0)$ to $1 \in \mathcal{B}_1(0)$, and for $\mathcal{S}_2$ we map $12 \in \mathcal{A}_2(0)$ to $12 \in \mathcal{B}_2(0)$ (the permutation $21 \in \mathcal{S}_2$ does not start with an odd number, so we do not consider it).

Now suppose for $\mathcal{S}_{2n}$ we have a bijective map between $\mathcal{A}_{2n}(k)$ and $\mathcal{B}_{2n}(k)$ for $k = 0, 1, \ldots, n-1$, and suppose that a permutation $\sigma \in \mathcal{A}_{2n}(k)$ corresponds to a permutation $\pi \in \mathcal{B}_{2n}(k)$. Based on σ and π we will match $n+k+1$ permutations from $\mathcal{A}_{2n+1}(k)$ to $n+k+1$ permutations from $\mathcal{B}_{2n+1}(k)$ (case A1 below), as well as $n-k$ permutations from $\mathcal{A}_{2n+1}(k+1)$ to $n-k$ permutations from $\mathcal{B}_{2n+1}(k+1)$ (case A2 below). One can see that all the maps below are bijective and all the permutations we deal with start with an odd number.

- A1:
  - 1.1: Map $\sigma^{(2n)}$ to $\pi^{(2n)}$.
  - 1.2: Map $\sigma^{(i)}$ to $\pi^{(j)}$ if i is the m-th descent in σ and j is the m-th descent in π ($m = 1, 2, \ldots, k$).
  - 1.3: Map $\sigma^{(i)}$ to $\pi^{(j)}$ if σ_{i+1} is the m-th odd number in σ and π_{j+1} is the m-th even number in π ($m = 1, 2, \ldots, n$).
- A2: Map $\sigma^{(i)}$ to $\pi^{(j)}$ if σ_{i+1} is the m-th even number such that $i \notin \overrightarrow{Des}_E(\sigma)$ and π_{j+1} is the m-th odd number such that $j \notin \overrightarrow{Des}_O(\pi)$ ($m = 1, 2, \ldots, n-k$).

Note that cases A1 and A2 cover all possible insertions of $2n+1$ in σ and π.

Now suppose for $\mathcal{S}_{2n+1}$ we have a bijective map between $\mathcal{A}_{2n+1}(k)$ and $\mathcal{B}_{2n+1}(k)$ for $k = 0, 1, \ldots, n$, and suppose that a permutation $\sigma \in \mathcal{A}_{2n+1}(k)$ corresponds to a permutation

$\pi \in \mathcal{B}_{2n+1}(k)$. Based on σ and π we will match $n+k+1$ permutations from $\mathcal{A}_{2n+1}(k)$ to $n+k+1$ permutations from $\mathcal{B}_{2n+1}(k)$ (case B1 below), as well as $n-k$ permutations from $\mathcal{A}_{2n+1}(k+1)$ to $n-k$ permutations from $\mathcal{B}_{2n+1}(k+1)$ (case B2 below).

B1: 1.1: Map $\sigma^{(2n+1)}$ to $\pi^{(2n+1)}$.

1.2: Map $\sigma^{(i)}$ to $\pi^{(j)}$ if i is the m-th descent in σ and j is the m-th descent in π ($m = 1, 2, \ldots, k$).

1.3: Map $\sigma^{(i)}$ to $\pi^{(j)}$ if σ_{i+1} is the $(m+1)$-st odd number in σ and π_{j+1} is the m-th even number in π ($m = 1, 2, \ldots, n$).

B2: Map $\sigma^{(i)}$ to $\pi^{(j)}$ if σ_{i+1} is the m-th even number such that $i \notin \overrightarrow{Des}_E(\sigma)$ and π_{j+1} is the $(m+1)$-st odd number such that $j \notin \overrightarrow{Des}_O(\pi)$ ($m = 1, 2, \ldots, n-k$).

Note that cases B1 and B2 cover all possible insertions of $2n+2$ in σ and π, and this finishes the construction of our bijective map.

8.5 Bijective (recursive) proof for $P_{1,k,n} = Q_{0,k+1,n}$.

The bijection we provide in this subsection is based on the (recursive) bijection (let us denote it by α) for $P_{0,k,n} = Q_{1,k,n}$ from Subsection 8.4. Using n-permutations σ and π, where $\sigma = \alpha(\pi)$, σ and π start with odd numbers, and $\overrightarrow{des}_E(\sigma) = \overrightarrow{des}_O(\pi) = k$, we will match k permutations counted by $P_{1,k-1,n+1}$ with k permutations counted by $Q_{0,k,n+1}$, as well as $n-k$ permutations counted by $P_{1,k,n+1}$ with $n-k$ permutations counted by $Q_{0,k+1,n+1}$. It will be clear that following our procedure, different σ and π such that $\sigma = \alpha(\pi)$, produce different pairs of permutations matched. Also, it will be easy to see that each $(n+1)$-permutation starting with an even number will be taken into account.

Given σ and π with the properties as above, we consider $2n$ permutations of $\{0, 1, \ldots, n\}$ obtained by placing the (even) number 0 in σ and π in different positions but position 0. Inserting 0 in σ in position i, where $i \in \overrightarrow{Des}_E(\sigma)$, produces σ' with $\overrightarrow{des}_E(\sigma') = k$. Inserting 0 in π in position i, where $i \in \overrightarrow{Des}_O(\pi)$, produces π' with $\overrightarrow{des}_O(\pi') = k-1$. We match such σ' and π' if the insertion was inside the m-th descent in both cases ($m = 1, 2, \ldots, k$).

Inserting 0 in a non-descent position in σ (resp. π) gives σ' (resp. π') with $\overrightarrow{des}_E(\sigma') = k+1$ (resp. $\overrightarrow{des}_O(\pi') = k$). We match such σ' and π' if the insertion in them was in the same (non-descent) position m counting from left to right ($m = 1, 2, \ldots, n-k$).

We now increase each number in all the σ' and π' considered above by 1 (evens become odds and vice versa) to get permutations counted by $Q_{0,k,n+1}$ been matched to permutations counted by $P_{1,k-1,n+1}$, as well as permutations counted by $Q_{0,k+1,n+1}$ been matched to permutations counted by $P_{1,k,n+1}$.

To summarize, our bijection works as follows. Given an $(n+1)$-permutation δ starting with an even number and having, say, $\overrightarrow{des}_E(\delta) = k$, we decrease each number of δ by 1 to get δ', a permutation of $\{0, 1, \ldots, n\}$ which starts with an odd number. If we now ignore 0 in δ', then we get an n-permutation δ'' starting with an odd number. We then let

$\tau'' = \alpha(\delta'')$. Depending on the position of 0 in δ' we insert 0 into τ'' to get a permutation τ' of $\{0, 1, \ldots, n\}$ which starts with an odd number and has the right properties (see instructions above). Finally, we increase all the numbers in τ' by 1 to get τ starting with an even number and having $\overrightarrow{des}_O(\tau) = k + 1$. Thus, τ corresponds to δ. The reverse to this is similar.

9 Conclusions

This paper can be viewed as a first step in a more general program which is to study pattern avoiding conditions permutations where generalized parity considerations are taking into account. Below we provide a possible parity generalization.

For any sequence of distinct numbers, $i_1 \cdots i_m$, we let $red(i_1 \cdots i_m)$ denote the permutation of $\mathcal{S}_m$ whose elements have the same relative order as $i_1 \cdots i_m$. For example, $red(5\ 2\ 7\ 8) = 2\ 1\ 3\ 4$. Then given a permutation $\tau = \tau_1 \cdots \tau_m \in \mathcal{S}_m$, we say that the permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$ is τ-avoiding if there is no subsequence $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_m}$ of σ such that $red(\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_m}) = \tau$. Similarly, we say that σ has a τ-match if there is a consecutive subsequence $\sigma_{i_1}\sigma_{i_1+1} \cdots \sigma_{i_1+m-1}$ such that $red(\sigma_{i_1}\sigma_{i_1+1} \cdots \sigma_{i_1+m-1}) = \tau$. There have been many papers in the literature that have studied the number of τ-avoiding permutations of $\mathcal{S}_n$ or the distribution of τ-matches for $\mathcal{S}_n$ (e.g., see [2, 6, 7] and references therein). Now we can generalize the notion of τ-avoiding permutations or τ-matches by adding parity type conditions. For example, for any integer $k \geq 2$, we say that a permutation is parity-k-τ-avoiding if there is no subsequence $\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_m}$ of σ such that $red(\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_m}) = \tau$ and for all j, $\sigma_{i_j} \equiv \tau_j \mod k$. Similarly, we say that σ has a parity-k-τ-match if there is a consecutive subsequence $\sigma_{i_1}\sigma_{i_1+1} \cdots \sigma_{i_1+m-1}$ such that $red(\sigma_{i_1}\sigma_{i_1+1} \cdots \sigma_{i_1+m-1}) = \tau$ and for all j, $\sigma_{i_j} \equiv \tau_j \mod k$. For example, the permutation $\sigma = 3\ 2\ 4\ 5\ 1$ is not 2 1-avoiding and has two 2 1-matches. However it is parity-2-2 1 avoiding and, hence, it has no parity-2-2 1-matches. Similarly, σ has parity-3-2 1 match since $red(5\ 1) = 2\ 1$, $5 \equiv 2 \mod 3$, and $1 \equiv 1 \mod 3$.

We have started to study such generalized parity matching type conditions. For example, in [8], we have generalized the results of this paper to classify descents according to equivalence mod k for $k \geq 3$. In [9], Liese and Remmel have studied the distribution of parity-k-τ matches for $\tau \in \mathcal{S}_2$.

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