On a pattern avoidance condition for the wreath product of cyclic groups with symmetric groups.

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Abstract

In this paper, we extend, to a non-consecutive case, the study of the pattern matching condition on the wreath product $C_k \wr S_n$ of the cyclic group C_k and the symmetric group S_n initiated in [2]. The main focus of our paper is (colored) patterns of length 2, although a number of enumerative results for longer patterns are also presented. A new non-trivial bijective interpretation for the Catalan numbers is found, which is the number of elements in $C_2 \wr S_n$ bi-avoiding simultaneously (1-2,0 0) and (1-2,0 1).

1 Introduction

The goal of this paper is to continue the study of pattern matching conditions on the wreath product $C_k \wr S_n$ of the cyclic group C_k and the symmetric group S_n initiated in [2]. $C_k \wr S_n$ is the group of $k^n n!$ signed permutations where there are k signs, $1 = \omega^0$, ω, ω^2, $\ldots$, ω^{k-1} where ω is a primitive k-th root of unity. We can think of the elements $C_k \wr S_n$ as pairs $\gamma = (\sigma, \epsilon)$ where $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ and $\epsilon = \epsilon_1 \cdots \epsilon_n \in \{1, \omega, \ldots, \omega^{k-1}\}^n$. For ease of notation, if $\epsilon = (\omega^{w_1}, \omega^{w_2}, \ldots, \omega^{w_n})$ where $w_i \in \{0, \ldots, k-1\}$ for $i = 1, \ldots, n$, then we simply write $\gamma = (\sigma, w)$ where $w = w_1 w_2 \cdots w_n$. Moreover, we think of the elements of $w = w_1 w_2 \cdots w_n$ as the *colors* of the corresponding elements of the underlying permutation σ.

Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let red(σ) be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then

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red(σ) = 1 4 3 2. Given a permutation τ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a *τ-match at place i* provided red($\sigma_i \cdots \sigma_{i+j-1}$) $= \tau$. Let τ-mch(σ) be the number of τ-matches in the permutation σ. Similarly, we say that τ *occurs* in σ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that red($\sigma_{i_1} \cdots \sigma_{i_j}$) $= \tau$. We say that σ *avoids* τ if there are no occurrences of τ in σ.

We can define similar notions for words over a finite alphabet $[k] = \{0, 1, \ldots, k-1\}$. Given a word $w = w_1 \cdots w_n \in [k]^*$, let red($w$) be the word found by replacing the i^{th} largest integer that appears in w by $i-1$. For example, if $w = 2\ 7\ 2\ 4\ 7$, then red(w) = 0 2 0 1 2. Given a word $u \in [k]^j$ such that red(u) $= u$, define a word $w \in [k]^n$ to have a *u-match at place i* provided red($w_i \cdots w_{i+j-1}$) $= u$. Let u-mch(w) be the number of u-matches in the word w. Similarly, we say that u *occurs* in a word w if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that red($w_{i_1} \cdots w_{i_j}$) $= u$. We say that w *avoids* u if there are no occurrences of u in w.

There are a number of papers on pattern matching and pattern avoidance in $C_k \wr S_n$ (see [1, 2, 3, 4, 5]). For example, the following pattern matching condition was studied in [3, 4, 5].

Definition 1. *1. We say that an element $(\tau, u) \in C_k \wr S_j$* **occurs** *in an element $(\sigma, w) \in C_k \wr S_n$ if there are $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ such that* red($\sigma_{i_1} \ldots \sigma_{i_j}$) $= \tau$ *and $w_{i_p} = u_p$ for $p = 1, \ldots, j$.*

2. We say that an element $(\sigma, w) \in C_k \wr S_n$ **avoids** *$(\tau, u) \in C_k \wr S_j$ if there are no occurrences of (τ, u) in (σ, w).*

3. If $(\sigma, w) \in C_k \wr S_n$ and $(\tau, u) \in C_k \wr S_j$, then we say that there is a (τ, u)-**match in** *(σ, w)* **starting at position** *i if* red($\sigma_i \sigma_{i+1} \ldots \sigma_{i+j-1}$) $= \tau$ *and $w_{i+p-1} = u_p$ for $p = 1, \ldots, j$.*

That is, an occurrence or match of $(\tau, u) \in C_k \wr S_j$ in an element $(\sigma, w) \in C_k \wr S_n$ is just an ordinary occurrence or match of τ in σ where the corresponding signs agree exactly. For example, Mansour [4] proved via recursion that for any $(\tau, u) \in C_k \wr S_2$, the number of (τ, u)-avoiding elements in $C_k \wr S_n$ is $\sum_{j=0}^{n} j!(k-1)^j \binom{n}{j}^2$. This generalized a result of Simion [6] who proved the same result for the hyperoctrahedral group $C_2 \wr S_n$. Similarly, Mansour and West [5] determined the number of permutations in $C_2 \wr S_n$ that avoid all possible 2 and 3 element set of patterns of elements of $C_2 \wr S_2$. For example, let K_n^1 be the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all the patterns in the set $\{(1\ 2, 0\ 0), (1\ 2, 0\ 1), (2\ 1, 1\ 0)\}$, K_n^2 be the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all the patterns in the set $\{(1\ 2, 0\ 1), (1\ 2, 1\ 0), (2\ 1, 0\ 1)\}$, and K_n^3 be the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all the patterns in the set $\{(1\ 2, 0\ 0), (1\ 2, 0\ 1), (2\ 1, 0\ 0)\}$. They proved that

$$\begin{aligned} K_n^1 &= F_{2n+1}, \\ K_n^2 &= n! \sum_{j=0}^{n} \binom{n}{j}^{-1}, \text{ and} \\ K_n^3 &= n! + n! \sum_{j=1}^{n} \frac{1}{j} \end{aligned}$$

where F_n is the n-th Fibonacci number.

In this paper, we shall consider the following pattern matching conditions which were first considered in [2].

Definition 2. *Suppose that* $(\tau, u) \in C_k \wr S_j$ *and* $\mathrm{red}(u) = u$.

1. *We say that* (τ, u) **bi-occurs** *in* $(\sigma, w) \in C_k \wr S_n$ *if there are* $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ *such that* $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$ *and* $\mathrm{red}(w_{i_1} \cdots w_{i_j}) = u$.

2. *We say that an element* $(\sigma, w) \in C_k \wr S_n$ **bi-avoids** (τ, u) *if there are no bi-occurrences* (τ, u) *in* (σ, w).

3. *We say that there is a* (τ, u)**-bi-match in** $(\sigma, w) \in C_k \wr S_n$ **starting at position** i *if* $\mathrm{red}(\sigma_i \sigma_{i+1} \ldots \sigma_{i+j-1}) = \tau$ *and* $\mathrm{red}(w_i w_{i+1} \ldots w_{i+j-1}) = u$.

One can easily extend these notions to sets of elements of $C_k \wr S_j$. That is, suppose that $\Upsilon \subseteq C_k \wr S_j$ is such that every $(\tau, u) \in \Upsilon$ has the property that $\mathrm{red}(u) = u$. Then (σ, w) has an Υ-bi-match at place i provided $(\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}), \mathrm{red}(w_i \cdots w_{i+j-1})) \in \Upsilon$, (σ, w) has a bi-occurrence of Υ if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $(\mathrm{red}(\sigma_{i_1} \ldots \sigma_{i_j}), \mathrm{red}(w_{i_1} \ldots w_{i)j})) \in \Upsilon$, and (σ, w) bi-avoids Υ if there is no bi-occurrence of Υ in (σ, w).

We call the pair (τ, u) in Definition 2 a *pattern*. Moreover, to distinguish patterns in case of bi-occurrences and bi-matches, we use for the former one dashes between the elements of τ. Thus, absence of a dash between any consecutive pair of elements, say x and y, in τ means that the elements in $(\sigma, w) \in C_k \wr S_n$ corresponding to x and y are adjacent. For example, the pattern (1 2,0 0) bi-occurs in $(\sigma, w) =$ (1 3 2 4,1 2 2 2) at position 3, whereas the pattern (1-2,0 0) bi-occurs twice in (σ, w) additionally involving the elements σ_3 and σ_4.

There are a number of natural maps which show that the problem of finding the distribution of bi-matches or bi-occurrences for various patterns is the same. That is, for any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we define the reverse of σ, σ^r, and the complement of σ, σ^c, by

$$\begin{aligned} \sigma^r &= \sigma_n \sigma_{n-1} \cdots \sigma_1 \text{ and} \\ \sigma^c &= (n+1-\sigma_1) \cdots (n+1-\sigma_n). \end{aligned}$$

Similarly, if $w = w_1 \cdots w_n \in [k]^n$, then we define the reverse of w, w^r, and the complement of w, w^c, by

$$\begin{aligned} w^r &= w_n w_{n-1} \cdots w_1 \text{ and} \\ w^c &= (k-1-w_1) \cdots (k-1-w_n). \end{aligned}$$

We can then consider maps $\phi_{a,b} : C_k \wr S_n \to C_k \wr S_n$ where $\phi_{a,b}((\sigma, w)) = (\sigma^a, w^b)$ for $a, b \in \{r, c\}$. It is easy to see that (σ, τ) has a (τ, w) bi-match or bi-occurrence if and only (σ^a, w^b) has a bi-match or bi-occurrence of (τ^a, u^b).

Study of (Υ-)bi-matches for patterns of length 2, i.e. where $(\tau, u) \in C_k \wr S_2$ was the main focus of [2]. Such bi-matches are closely related to the analogue of rises and descents in $C_k \wr S_n$ where we compare pairs using the product order. That is, instead of thinking of an element of $C_k \wr S_n$ as a pair $(\sigma_1 \cdots \sigma_n, w_1 \cdots w_n)$, we can think of it as a sequence of pairs $(\sigma_1, w_1)(\sigma_2, w_2) \cdots (\sigma_n, w_n)$. We then define a partial order on such pairs by the usual product order. That is, $(i_1, j_1) \leq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$.

The outline of this paper is as follows. In section 2, we shall consider bi-avoidance for patterns of length 2. Up to equivalence of the maps induced by complementation and reversal, there are only 2 classes of patterns, namely those which are equivalent to (1-2,0 0) and those that are equivalent to (1-2,0 1). For patterns of the form $(\tau, 0^j)$, we derive formulas for the number of elements of $C_k \wr S_n$ which bi-avoid $(\tau, 0^j)$ from the formulas for the number of elements of S_n which avoid τ, or the distribution of the number of bi-occurrences of $(\tau, 0^j)$ in elements of $C_k \wr S_n$ from the corresponding formulas for the distribution of occurrences of τ in S_n. For the pattern (1-2,0 1), we can only find a formula for the number of elements of $C_2 \wr S_n$ which bi-avoid (τ, u). In section 3, we shall derive formulas for the number of elements of $C_k \wr S_n$ which bi-avoid various sets of patterns of length 2. In particular, we prove bijectively (see Theorem 13) that the number of permutations in $C_2 \wr S_n$ simultaneously bi-avoiding (1-2,0 0) and (1-2,0 1) is given by the $(n+1)$-th Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$.

2 Bi-avoiding patterns

In this section, we shall consider bi-avoidance with respect to patterns of length 2. Up to the equivalence induced by the reversal and complement maps for permutations and words, we need only consider 2 patterns, (1-2,0 0) and (1-2,0 1).

We start by considering the pattern (1-2, 0 0).

Theorem 3. *The number of permutations in $C_k \wr S_n$ bi-avoiding (1-2,0 0) is given by*

$$\sum_{\substack{i_1 + \cdots + i_k = n \\ i_1 \geq 0, \ldots, i_k \geq 0}} \binom{n}{i_1, \ldots, i_k}^2.$$

Proof. We first observe that elements of different colors are independent in permutations bi-avoiding (1-2,0 0), meaning that no two elements with different colors can form a prohibited configuration. Thus, assuming we have i_{j+1} elements of color j, $j = 0, \ldots, k-1$, we can choose how to place these colors to form a word w in $\binom{n}{i_1,\ldots,i_k}$ ways. Then we can choose the sets of elements $C_0, \ldots, C_{k-1}$ from $\{1, \ldots, n\}$ which will correspond to the colors $0, 1, \ldots, k-1$ in σ in $\binom{n}{i_1,\ldots,i_k}$ ways. Finally, in order to construct (σ, w) which bi-avoids the prohibited pattern, we must place the elements of C_i in the positions which are colored i in decreasing order. □

The proof of Theorem 3 suggests an obvious generalization for patterns of arbitrary length with all possible dashes whose elements are all colored by 0:

Theorem 4. *Let p be a pattern containing dashes in all places. The number of permutations in $C_k \wr S_n$ bi-avoiding $(p, 0 \cdots 0)$ is given by*

$$\sum_{\substack{i_1 + \cdots + i_k = n \\ i_1 \geq 0, \ldots, i_k \geq 0}} A_{i_1} A_{i_2} \cdots A_{i_k} \binom{n}{i_1, \ldots, i_k}^2$$

where A_i is the number of permutations in S_i avoiding p.

Proof. A proof here is essentially the same as the proof of Theorem 3, except we can place i elements of a permutation in $C_k \wr S_n$ of the same color in any of A_i ways. □

It is well known that the number of n-permutations avoiding any pattern of length 3 with dashes everywhere is given by the *n-th Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. As a corollary to Theorem 4, we have that for any pattern p of length 3 with two dashes, the number of permutations in $C_k \wr S_n$ bi-avoiding (p,0 0 0) is given by

$$\sum_{\substack{i_1+\cdots+i_k=n\\ i_1\geq 0,\ldots,i_k\geq 0}} \frac{\binom{2i_1}{i_1}\binom{2i_2}{i_2}\cdots\binom{2i_k}{i_k}}{(i_1+1)(i_2+1)\cdots(i_k+1)}\binom{n}{i_1,\ldots,i_k}^2.$$

One can generalize even further Theorems 3 and 4, by talking on distribution of patterns. Indeed, assuming we know the number $A_{i,j}$ of i-permutations containing j occurrences of a pattern p (with dashes everywhere), we can write down the number of permutations in $C_k \wr S_n$ with j bi-occurrence of (p,$0\cdots 0$):

$$\sum_{\substack{i_1+\cdots+i_k=n\\ j_1+\cdots+j_k=j\\ i_1\geq 0,\ldots,i_k\geq 0\\ j_1\geq 0,\ldots,j_k\geq 0}} A_{i_1,j_1}A_{i_2,j_2}\cdots A_{i_k,j_k}\binom{n}{i_1,\ldots,i_k}^2.$$

For example, the distribution of the pattern 1-2 is the same as the distribution of *inversions* in permutations (which coincides with the distribution of 2-1), so one can extract the numbers $A_{i,j}$ as the coefficients to q^j in $\prod_{m=1}^{i}(1+q+\cdots+q^{m-1})$ and substitute them in the last formula to get the distribution of the number of bi-occurrences of (1-2,0 0) in $C_k \wr S_n$.

Next we consider the pattern (1-2,0 1) in the case where $k=2$.

The number of permutations in $C_2 \wr S_n$ bi-avoiding (1-2,0 1) is shown in [6, page 19] to be equal to $\sum_{j=0}^{n} j!\binom{n}{j}^2$. However, in Theorem 5 below we provide an independent derivation of the exponential generating function in this case.

Theorem 5. *The exponential generating function for the number of permutations in $C_2 \wr S_n$ bi-avoiding the pattern (1-2,0 1) is given by*

$$A(x)=\frac{e^{\frac{x}{1-x}}}{1-x}.$$

Proof. Let A_n denote the number of n-permutations in $C_2 \wr S_n$ bi-avoiding the pattern (1-2,0 1). If an $(n+1)$-permutation contains the element 1 colored by the color 1, then there are no restrictions for placing this element, thus giving $(n+1)A_n$ possibilities. On the other hand, if an $(n+1)$-permutation (σ,w) contains the element 1 colored by the color 0 in position $i+1$, then if (σ,w) is to bi-avoid (1-2,0 1), then every element to the right of 1 must be colored with color 0 and these elements can be arranged in any of $(n-i)!$ ways. Moreover, it immediately follows that no instance of a bi-occurrence of (1-2,0 1) exists where the first element is to the left of 1, and the second is to the right of 1. Thus it follows that (σ,w) bi-avoids (1-2,0 1) if and only if there is no bi-occurrence of (1-2,0 1) in $(\sigma_1\cdots\sigma_i, w_1\cdots w_i)$. Thus, in the case where 1 colored by 0 and is in position $i+1$, we have $(n-i)!\binom{n}{i}A_i = n!A_i/i!$ possibilities where the binomial coefficient is responsible for choosing the elements to the left of 1, and placing the remaining elements to the right of 1. To summarize, we obtain

$$A_{n+1}=n!\sum_{i=0}^{n}\frac{A_i}{i!}+(n+1)A_n.$$

Multiplying both parts of the equation above by $x^n/n!$ and summing over all $n\geq 1$, we have

$$-A_1+A'(x)=-A_0+A(x)/(1-x)+xA'(x)+A(x)-A_0$$

leading to the differential equation

$$A'(x) = \frac{2-x}{(1-x)^2} A(x)$$

with the initial condition $A_0 = 1$ as the empty permutation bi-avoids (1-2,0 1). The solution to this differential equation is

$$A(x) = \frac{e^{\frac{x}{1-x}}}{1-x}.$$

□

3 Bi-avoidance for sets of patterns

In this section, we shall prove a variety of results for the number of elements of $C_k \wr S_n$ that bi-avoid certain sets of patterns of length 2. For any set $\Upsilon \subseteq C_k \wr S_j$ such that $red(u) = u$ for all $(\tau, u) \in \Upsilon$, we let $Av_{n,k}^{\Upsilon}$ denote the number of elements of $C_k \wr S_n$ which bi-avoid Υ.

We start with a few simple results on sets of patterns Υ where the bi-avoidance of Υ forces certain natural conditions on the possible sets of signs for elements of $C_k \wr S_n$.

Theorem 6.

1. *If* $\Upsilon_1 = \{(\text{1-2}, 0\ 0), (\text{2-1}, 0\ 0)\}$, *then* $Av_{n,k}^{\Upsilon_1} = \binom{k}{n} n!n!$ *for all* $n \geq 1$ *and* $k \geq 1$.
2. *If* $\Upsilon_2 = \{(\text{1-2}, 1\ 0), (\text{2-1}, 1\ 0)\}$, *then* $Av_{n,k}^{\Upsilon_2} = \binom{n+k-1}{n} n!$ *for all* $n \geq 1$ *and* $k \geq 1$.
3. *If* $\Upsilon_3 = \{(\text{1-2}, 0\ 0), (\text{1-2}, 1\ 0), (\text{2-1}, 0\ 0), (\text{2-1}, 1\ 0)\}$, *then* $Av_{n,k}^{\Upsilon_3} = \binom{k}{n} n!$ *for all* $n \geq 1$ *and* $k \geq 1$.
4. *If* $\Upsilon_4 = \{(\text{1-2}, 0\ 1), (\text{1-2}, 1\ 0), (\text{2-1}, 0\ 1), (\text{2-1}, 1\ 0)\}$, *then* $Av_{n,k}^{\Upsilon_3} = kn!$ *for all* $n \geq 1$ *and* $k \geq 1$.

Proof. For (1), it is easy to see that $(\sigma, w) \in C_k \wr S_n$ bi-avoids Υ_1 if and only if all the signs are pairwise distinct. Thus there are $\binom{k}{n}$ ways to pick the n signs and then you have $n!$ ways to arrange those n signs and $n!$ ways to pick σ.

For (2), it is easy to see that $(\sigma, w) \in C_k \wr S_n$ bi-avoids Υ_2 if and only $0 \leq w_1 \leq \cdots \leq w_n \leq k-1$. Thus there are $\binom{n+k-1}{n}$ ways to pick the n signs in this case and there are $n!$ ways to pick σ.

For (3), it is easy to see that $(\sigma, w) \in C_k \wr S_n$ bi-avoids Υ_2 if and only $0 \leq w_1 < \cdots < w_n \leq k-1$. Thus there are $\binom{k}{n}$ ways to pick the n signs in this case and there are $n!$ ways to pick σ.

For (4), it is easy to see that $(\sigma, w) \in C_k \wr S_n$ bi-avoids Υ_2 if and only $0 \leq w_1 = \cdots = w_n \leq k-1$. Thus there are k ways to pick the n signs in this case and there are $n!$ ways to pick σ. □

If $\Upsilon \subseteq S_j$ is a set of permutations of S_j, we let

$$\phi^{\Upsilon}(x,t) = \sum_{n \geq 0} \frac{t^n}{n!} Av_n^{\Upsilon} \tag{1}$$

where Av_n^{Υ} is the number of permutations $\sigma \in S_n$ such that σ avoids Υ. Similarly, if $\Upsilon \subseteq C_k \wr S_j$ is such that for all $(\sigma, u) \in \Upsilon$, $red(u) = u$, then we let

$$\theta_k^{\Upsilon}(x,t) = \sum_{n \geq 0} \frac{t^n}{n!} Av_{n,k}^{\Upsilon}. \tag{2}$$

We let $\Upsilon_0 = \{(\sigma, 0^j) : \sigma \in \Upsilon\}$. Then we have the following.

Theorem 7. *Let $\Upsilon \subseteq S_j$ be any set of permutations of S_j and $\Upsilon_0 = \{(\sigma, 0^j) : \sigma \in \Upsilon\}$. Then if $\Gamma = \Upsilon_0 \cup \{(\text{1-2}, 1\ 0), (\text{2-1}, 1\ 0)\}$,*

$$\theta_k^{\Gamma}(t) = (\phi^{\Upsilon}(t))^k \tag{3}$$

for all $k \geq 1$.

Proof. It is easy to see that if $(\sigma, u) \in C_k \wr S_n$ and (σ, u) bi-avoids $\{(\text{1-2}, 1\ 0), (\text{2-1}, 1\ 0)\}$, then it must be the case that $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq k-1$. Thus suppose that $w = 0^{a_1}1^{a_2} \cdots (k-1)^{a_k}$ where $a_1 + \cdots + a_k = n$ and $a_i \geq 0$ for $1 \leq i \leq k$. Then clearly the number of $\sigma \in S_n$ such that (σ, w) bi-avoids Υ_0 is

$$\binom{n}{a_1, \ldots, a_k} Av_{a_1}^{\Upsilon} \cdots Av_{a_k}^{\Upsilon}.$$

That is, the binomial coefficient allows us to choose the elements C_i of $\{1, \ldots, n\}$ that correspond to the constant segment $i^{a_{i+1}}$ in w. Then we only have to arrange the elements of C_i so that it avoids Υ which can be done in $Av_{a_{i+1}}^{\Upsilon}$ ways. Thus it follows that

$$Av_{n,k}^{\Gamma} = \sum_{a_1 + \cdots + a_k = n; a_i \geq 0} \binom{n}{a_1, \ldots, a_k} Av_{a_1}^{\Upsilon} \cdots Av_{a_k}^{\Upsilon}.$$

or equivalently

$$\frac{Av_{n,k}^{\Gamma}}{n!} = \sum_{a_1 + \cdots + a_k = n; a_i \geq 0} \prod_{i=1}^{k} \frac{Av_{a_i}^{\Upsilon}}{a_i!}$$

which implies (3). □

We immediately have the following corollary.

Corollary 8. *For any $k \geq 1$, the number of elements of $C_k \wr S_n$ which bi-avoids $\Gamma_1 = \{(\text{1-2}, 0\ 0), (\text{1-2}, 1\ 0), (\text{2-1}, 1\ 0)\}$ or $\Gamma_2 = \{(\text{2-1}, 0\ 0), (\text{1-2}, 1\ 0), (\text{2-1}, 1\ 0)\}$ is k^n.*

Proof. Let $\Upsilon_1 = \{1\ 2\}$ and $\Upsilon_2 = \{2\ 1\}$. Then clearly, $Av_n^{\Upsilon_i} = 1$ for $i = 1, 2$ so that $\phi_n^{\Upsilon_i}(t) = e^t$ for $i = 1, 2$. But then by Theorem 7, $\theta_{n,k}^{\Gamma_i}(t) = (e^t)^k = e^{kt}$ so that $Av_{n,k}^{\Gamma_i} = k^n$ for $i = 1, 2$. □

We can derive theorems analogous to Theorem 7 for the other sign conditions in Theorem 6. That is, suppose that Υ is any set of patterns contained in S_j. Then we let

$$\Upsilon_i = \{(\tau, 0\ 1 \ldots (j-1)) : \tau \in \Upsilon\}$$

and

$$\Upsilon_d = \{(\tau, u) : \tau \in \Upsilon \ \&\ u \in D_j\}$$

where D_j is the set of all permutations of $\{0, 1, \ldots, j-1\}^*$. Then we have the following.

Theorem 9. *For any $\Upsilon \subset S_j$, let*

$$\begin{aligned}\Gamma_1 &= \Upsilon_0 \cup \{(1\text{-}2, 0\ 1), (1\text{-}2, 1\ 0), (2\text{-}1, 0\ 1), (2\text{-}1, 1\ 0)\}, \\ \Gamma_2 &= \Upsilon_i \cup \{(1\text{-}2, 1\ 0), (1\text{-}2, 0\ 0), (2\text{-}1, 1\ 0), (2\text{-}1, 0\ 0)\}, \text{ and} \\ \Gamma_3 &= \Upsilon_d \cup \{(1\text{-}2, 0\ 0), (2\text{-}1, 0\ 0)\}.\end{aligned}$$

Then for all $k \geq 1$,

1. $Av^{\Gamma_1}_{n,k} = kAv^{\Upsilon}_n$ *for all $n \geq 1$,*

2. $Av^{\Gamma_2}_{n,k} = \binom{k}{n} Av^{\Upsilon}_n$ *for all $n \geq 1$, and*

3. $Av^{\Gamma_2}_{n,k} = \binom{k}{n} n! Av^{\Upsilon}_n$ *for all $n \geq 1$.*

Proof. For (1), note that for $(\sigma, w) \in C_k \wr S_n$ to bi-avoid $\{(1\text{-}2, 0\ 1), (1\text{-}2, 1\ 0), (2\text{-}1, 0\ 1), (2\text{-}1, 1\ 0)\}$, $w = i^n$ for some $i \in \{0, \ldots, k-1\}$. Then $(\sigma, i^n) \in C_k \wr S_n$ bi-avoids Υ_0 if and only if σ avoids Υ. Thus $Av^{\Gamma_1}_{n,k} = kAv^{\Upsilon}_n$.

For (2), note that for $(\sigma, w) \in C_k \wr S_n$ to bi-avoid $\{(1\text{-}2, 1\ 0), (1\text{-}2, 0\ 0), (2\text{-}1, 1\ 0), (2\text{-}1, 0\ 0)\}$, $w = w_1 \cdots w_n$ where $w_1 < \cdots < w_n$. Then for any of the $\binom{k}{n}$ strictly increasing words $w \in \{0, \ldots, k-1\}$, $(\sigma, w) \in C_k \wr S_n$ bi-avoids Υ_i if and only if σ avoids Υ. Thus $Av^{\Gamma_2}_{n,k} = \binom{k}{n} Av^{\Upsilon}_n$.

For (3), note that for $(\sigma, w) \in C_k \wr S_n$ to bi-avoid $\{(1\text{-}2, 0\ 0), (2\text{-}1, 0\ 0)\}$, $w = w_1 \cdots w_n$ where the letters of w are pairwise distinct. Then for any of the $\binom{k}{n} n!$ words $w \in \{0, \ldots, k-1\}$ which have pairwise distinct letters, $(\sigma, w) \in C_k \wr S_n$ bi-avoids Υ_d if and only if σ avoids Υ. Thus $Av^{\Gamma_3}_{n,k} = \binom{k}{n} n! Av^{\Upsilon}_n$. ☐

Next we will prove two more results about $Av^{\Upsilon}_{n,k}$ for other sets of patterns Υ that contain $\{(1\text{-}2, 1\ 0), (2\text{-}1, 1\ 0)\}$

Theorem 10. *Let*

$$\begin{aligned}\Upsilon_1 &= \{(1\text{-}2, 0\ 1), (1\text{-}2, 1\ 0), (2\text{-}1, 1\ 0)\} \text{ and} \\ \Upsilon_2 &= \{(1\text{-}2, 0\ 1), (1\text{-}2, 1\ 0), (2\text{-}1, 1\ 0), (2\text{-}1, 0\ 0)\}.\end{aligned}$$

Then

1. $Av^{\Upsilon_1}_{n,k} = \sum_{a_1+\cdots+a_k=n, a_i \geq 0} a_1! \cdots a_k!$ *for all $n \geq 1$ and $k \geq 1$ and*

2. $Av^{\Upsilon_1}_{n,k} = \binom{n+k-1}{k-1}$ *for all $n \geq 1$ and $k \geq 1$.*

Proof. For (1), note that as in the proof of Theorem 7, if $(\sigma, u) \in C_k \wr S_n$ and (σ, u) bi-avoids $\{(1\text{-}2, 1\ 0), (2\text{-}1, 1\ 0)\}$, then it must be the case that $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq k-1$.

Now suppose that $w = 0^{a_1} 1^{a_2} \cdots (k-1)^{a_k}$ where $a_1 + \cdots + a_k = n$. Assume that (σ, w) also bi-avoids (1-2,0 1) and C_i is the set of elements of σ that correspond to the signs $(i-1)^{a_i}$ in w. Then it follows that all the elements of C_1 must be bigger than all the elements of C_2, all the elements of C_2 must be bigger than all the elements of C_3, etc.. Thus C_1 consists of the a_1 largest elements of $\{1, \ldots, n\}$, C_2 consists of next a_2 largest elements of $\{1, \ldots, n\}$, etc.. Then we can arrange the elements of C_i in any order in the positions corresponding to $(i-1)^{a_i}$ in w

to produce a (σ, w) which bi-avoids Υ. Hence there are $a_1! \cdots a_k!$ elements of the form (σ, w) which bi-avoid Υ in $C_k \wr S_n$. Thus (1) immediately follows.

For (2), observe that if in addition such (σ, w) also avoids (2-1,0 0), then we must place the elements of C_i in increasing order. Hence $Av_{n,k}^{\Upsilon_2}$ is the number of solutions of $a_1 + \cdots + a_k = n$ with $a_i \geq 0$ which is well known to be $\binom{n+k-1}{k-1}$. □

Theorem 11. *Let* $\Upsilon = \{(1\text{-}2, 0\ 1), (1\text{-}2, 1\ 0), (2\text{-}1, 0\ 0)\}$. *Then for* $n \geq 1$,

$$\begin{aligned} Av_{n,1}^{\Upsilon} &= 1, \\ Av_{n,2}^{\Upsilon} &= 2n, \text{ and} \\ Av_{n,k}^{\Upsilon} &= k + \sum j = 2^{k-1}(k) \downarrow_j \binom{n}{j} \text{ for } k \geq 3, \end{aligned}$$

where $(k) \downarrow_0 = 1$ *and* $(k) \downarrow_n = k(k-1) \cdots (k-n+1)$ *for* $n \geq 1$.

Proof. We shall classify the elements $(\sigma, w) \in C_k \wr S_n$ which bi-avoid Υ by the number of elements s which follow n in σ. Now if $s = 0$ so that σ ends in n, the fact that (σ, w) bi-avoids both (1-2,0 1) and (1-2,1 0) means that all the signs must be the same. That is, w must be of the form i^n for some $i \in [k]$. But since (σ, w) must also bi-avoid (2-1,0 0), $\sigma = 1\ 2 \cdots (n-1)\ n$ must be the identity. Thus there are k choices for such (σ, w).

Now suppose there are s elements following n in σ. Then again the fact that (σ, w) bi-avoids both (1-2,0 1) and (1-2,1 0) means that all the signs in w corresponding to $\sigma_1 \cdots \sigma_{n-s} = n$ must be the same, say that sign is i. The fact that (σ, w) avoids (2-1,0 0) means that (i) $\sigma_1 \cdots \sigma_{n-s}$ must be in increasing order and (ii) all the signs corresponding to $\sigma_{n-s+1} \ldots \sigma_n$ must be different from i. But then it follows from the fact that (σ, w) avoids both (1-2,0 1) and (1-2,1 0) that all the elements in $\sigma_1 \ldots \sigma_{n-s}$ must be greater than all the elements in $\sigma_{n-s+1} \cdots \sigma_n$. Hence there are $kAv_{s,k-1}^{\Upsilon}$ such elements if $k \geq 2$ and there are no such elements if $k = 1$. Thus it follows that $Av_{n,1}^{\Upsilon} = 1$ since $(1\ 2\ \cdots n, 0^n)$ is the only element of $C_k \wr S_n$ which bi-avoids Υ. For $k \geq 2$, we have

$$Av_{n,k}^{\Upsilon} = k + \sum_{s=1}^{n-1} kAv_{s,k-1}^{\Upsilon}. \tag{4}$$

In the case $n = 2$, (4) becomes

$$Av_{n,2}^{\Upsilon} = 2 + \sum_{s=1}^{n-1} 2 = 2n. \tag{5}$$

and in the case $k = 3$, (4) becomes

$$Av_{n,2}^{\Upsilon} = 3 + \sum_{s=1}^{n-1} 3(2s) = 3 + (3 \cdot 2)\binom{n}{2}. \tag{6}$$

In general, assuming that

$$Av_{n,k}^{\Upsilon} = k + \sum_{j=2}^{k-1} (k) \downarrow_j \binom{n}{j},$$

it follows that

$$
\begin{aligned}
Av^{\Upsilon}_{n,k+1} &= k+1+\sum_{s=1}^{n-1}(k+1)Av^{\Upsilon}_{s,k} \\
&= k+1+\sum_{s=1}^{n-1}(k+1)\left(k+\sum_{j=2}^{k-1}(k)\downarrow_j \binom{s}{j}\right) \\
&= k+1+\left(\sum_{s=1}^{n-1}(k+1)k\right)+\sum_{j=2}^{k-1}(k+1)(k)\sum_{s=1}^{n-1}\binom{s}{j} \\
&= k+1+(k+1)\downarrow_2 \binom{n}{2}+\sum_{j=3}^{k}(k+1)\downarrow_j \binom{n}{j}.
\end{aligned}
$$

□

We next consider simultaneous bi-avoidance of the patterns (1-2, 1 0) and (1-2, 0 1).

Theorem 12. *Let* $\Upsilon = \{(1\text{-}2, 1\ 0), (1\text{-}2, 0\ 1)\}$ *and let* $A^{\Upsilon}_k(t) = \sum_{n\geq 0} Av^{\Upsilon}_{n,k} t^n$ *and* $C(t) = \sum_{n\geq 1} n! t^n$. *Then*

$$
A^{\Upsilon}_k(t) = \frac{1+C(t)}{1-(k-1)C(t)}. \tag{7}
$$

Proof. Fix k and suppose that (σ, w) is an element of $C_k \wr S_n$ which bi-avoids both (1-2, 1 0) and (1-2, 0 1). Now if $w = i^n$ is constant, then clearly σ can be arbitrary so that there are $n!$ such elements.

Next assume that w is not constant so there is an $s \geq 2$ such that $w = i_1^{a_1} i_2^{a_2} \cdots i_s^{a_s}$ where $a_r \geq 1$ for $r = 1, \ldots, s$ and $i_r \neq i_{r+1}$ for $r = 1, \ldots, s-1$. We claim that $\sigma_1 \cdots \sigma_{a_1}$ must be the a_1 largest elements of $\{1, \ldots, n\}$. If not, then let $n-r$ be the largest element which is not in $\sigma_1 \cdots \sigma_s$. Thus $r < a_1 - 1$ and there must be at least one σ_t with $t \leq s$ such that $\sigma_t < n-r$. Now it cannot be that $\sigma_{a_1+1} = n-r$ since otherwise $(\sigma_t\ \sigma_{a_1+1}, i_1\ i_2)$ would be an occurrence of either (1-2,1 0) or (1-2,0 1). Hence it must be the case that $\sigma_{a_1+1} < n-r$ and $\sigma_p = n-r$ for some $p > a_1 + 1$. But then no matter what color we choose for w_p, either $(\sigma_t\ \sigma_p, i_1\ w_p)$ or $(\sigma_{a_1+1}\ \sigma_p, i_2\ w_p)$ would be an occurrence of either (1-2,1 0) or (1-2,0 1). We can continue this reasoning to show that for any $1 \leq p < q \leq s$, the elements of σ corresponding to the block $i_p^{a_p}$ in w must be strictly larger than the elements of σ corresponding to the block $i_q^{a_q}$ in w. This given, it follows that we can arrange the elements of σ corresponding to a block $i_p^{a_p}$ in w in any way that we want and we will always produce a pair (σ, w) that bi-avoids both (1-2,1 0) and (1-2,0 1). Thus for such a w, we have $k(k-1)^{s-1}$ ways to choose the colors $i_1, \ldots, i_s$ and $a_1! a_2! \cdots a_k!$ ways to choose the permutation σ. It follows that

$$
Av^{\Upsilon}_{n,k} = k\ n! + \sum_{s=2}^{n-1} \sum_{a_1+\cdots+a_s=n, a_i>0} k(k-1)^{s-1} a_1! a_2! \cdots a_s! \tag{8}
$$

which is equivalent to (7). □

Finally, we end this section by considering the number of elements of $C_k \wr S_n$ which bi-avoid both (1-2,0 0) and (1-2,0 1). In this case, we only have a result for the case when $k = 2$.

Theorem 13. *The number of permutations in $C_2 \wr S_n$ simultaneously bi-avoiding (1-2,0 0) and (1-2,0 1) is given by the $(n+1)$-th Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$.*

Proof. We prove the statement by establishing a bijection between the objects in question of length n and the *Dyck paths* of semi-length $(n+1)$ known to be counted by the Catalan numbers (A Dyck path of semi-length n is a lattice path from (0,0) to $(2n, 0)$ with steps (1,1) and $(1, -1)$ that never goes below x-axis).

Suppose $(\sigma, w) \in C_2 \wr S_n$. Note that σ must bi-avoid the pattern 1-2-3, as in an occurrence of such pattern in σ, there are two letters of the same color leading to an occurrence of (1-2,0 0). Thus, the structure of σ, as it is well-known, is two decreasing sequences shuffled.

Subdivide σ into so called *reverse irreducible components*. A reverse irreducible component is a factor F of σ of minimal length such that everything to the left (resp. right) of F is greater (resp. smaller) than any element of F. For example, the subdivision of $\sigma = 6574312$ is $\sigma = 657 - 4 - 3 - 12$. The blocks of size 1 are *singletons*. In the example above, 4 and 3 are singletons. This is easy to see, that any singleton element in σ can have any color (either 0 or 1). We will now show that the color of each element of a non-singleton block is uniquely determined.

Indeed, irreducibility of a single block means that two decreasing sequences in the structure of (1-2-3)-avoiding permutations are the block's sequence of left-to-right minima and the block's sequence of right-to-left maxima which do *not* overlap. Thus, for any left-to-right minimum element x (except possibly the last element), one has an element y greater than x to the right of it inside the same block and vice versa, from which we conclude that x must receive color 1, whereas y must receive color 0 (otherwise a prohibited pattern will bi-occur).

We are ready to describe our bijection. For a given $(\sigma, w) \in C_2 \wr S_n$, consider the matrix representation of σ, that is an integer grid with the opposite corners in (0,0) and (n, n), and with a dot in position $(i - \frac{1}{2}, \sigma_i - \frac{1}{2})$ for $i = 1, 2, \ldots, n$. We will give a description of a path P (corresponding to (σ, w)) from $(0, n+1)$ to $(n+1, 0)$ involving only steps $(0, -1)$ and (1,0) that never goes above the line $y = -x + n + 1$. Clearly, P can be transformed to a Dyck path of length $(n+1)$ by taking a mirror image with respect to the line $y = -x + n + 1$, rotating 45 degrees counterclockwise, and making a parallel shift.

To build P, set $i := 0$ and $j := n + 1$, and do the following steps letting P begin at (i, j). Clearly, each reverse irreducible block of σ defines a square on the grid which is the matrix representation of the block. We call the reverse irreducible block of σ with the left-top corner at (i, j) the *current block*.

Step 1. If the current block is *not* a singleton, go to Step 2. If the color of the element with x-coordinate equal $i + \frac{1}{2}$ is 0 (resp. 1) travel around the current block counterclockwise (resp. clockwise) to get to the point $(i + 1, j - 1)$. Note that P touches the line $y = -x + n + 1$ if the color is 1. Set $i := i + 1$ and $j := j - 1$, and proceed with Step 3.

Step 2. In Step 2 we follow a standard bijection between (1-2-3)-avoiding permutations and Dyck paths that can be described as follows. Let (k, ℓ) be the point of the current block opposite to (i, j). Start going down from (i, j) until the y-coordinate of the current node gets $\frac{1}{2}$ less than the y-coordinate of the dot with x-coordinate equal $i + \frac{1}{2}$. Start moving horizontally to the right and go as long as possible making sure that none of the dots are below the part of P constructed so far and $i \leq k$ and $j \leq \ell$. Suppose (i_1, j_1) is the last point the procedure above can be done (that is, we were traveling on the line $y = j_1$ and either

$i_1 = k$ and $j_1 = \ell$ or there is a dot with x-coordinate $i_1 + \frac{1}{2}$ having y-coordinate less than j_1). If $i := k$ and $j := \ell$, proceed with Step 3; otherwise set $i := i_1$ and $j := j_1$ and go to Step 2. Note that in Step 2, P never touches the line $y = -x + n + 1$.

Step 3. If $j = 0$, make as many as it takes horizontal steps to get to the point $(n + 1, 0)$ and terminate; otherwise go to Step 1.

Returning to our example, $\sigma = (6\ 5\ 7\ 4\ 3\ 1\ 2, 1\ 1\ 0\ 1\ 0\ 1\ 0)$, we have given the matrix diagram and outlined the reverse irreducible blocks in Figure 1. We start our path at $(0, 8)$. We travel down until we reach $(0, 6)$, when we are $\frac{1}{2}$ less than the y-coordinate of our first point $(\frac{1}{2}, 6\frac{1}{2})$. We then continue traveling right and down as described in Step 2. We travel clockwise around our singleton colored 1, and counterclockwise around our singleton colored 0. Then we continue to the final reverse irreducible block and finish our path, given in Figure 2. The resulting path is presented in Figure 3.

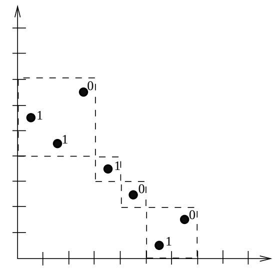

Figure 1: Matrix representation of $\sigma = (6\ 5\ 7\ 4\ 3\ 1\ 2, 1\ 1\ 0\ 1\ 0\ 1\ 0)$ with reverse irreducible blocks outlined.

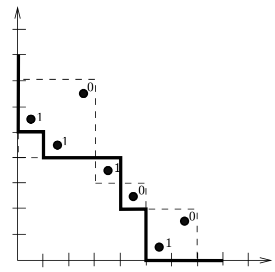

Figure 2: The Dyck path corresponding to $\sigma = (6\ 5\ 7\ 4\ 3\ 1\ 2, 1\ 1\ 0\ 1\ 0\ 1\ 0)$ added to Figure 1.

$\square$

4 Concluding remarks

A natural way to extend work in this paper is to consider bi-occurrences of patterns in $C_k \wr S_n$ of length more than 2. However, there are questions left regarding the pattern (1-2,0 1), which are listed below starting from the most ambitious one:

- Find distribution of (1-2,0 1) on $C_k \wr S_n$.

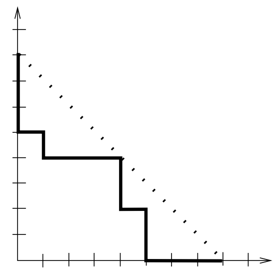

Figure 3: The Dyck path corresponding to $\sigma = (6\ 5\ 7\ 4\ 3\ 1\ 2, 1\ 1\ 0\ 1\ 0\ 1\ 0)$. Note the path only touches the line $y = -x + n + 1$ after a singleton colored 1.

- For $k \geq 3$, find the number of permutations in $C_k \wr S_n$ bi-avoiding (1-2,0 1). The number of permutations in $C_k \wr S_n$ bi-avoiding (1-2,0 1) for initial values of k and n are as follows:

$$\begin{array}{ll} k=3: & 3, 15, 101, 842, 8302, \ldots \\ k=4: & 4, 26, 224, 2361, \ldots \\ k=5: & 5, 40, 420, 5355, \ldots . \end{array}$$

- OEIS [7, A002720] suggests that the number of permutations in $C_2 \wr S_n$ bi-avoiding (1-2,0 1) (for initial values of n these numbers are 2, 7, 34, 209, 1546, 13327,...) is the same as the number of
  - partial permutations of an n-set;
  - $n \times n$ binary matrices with at most one 1 in each row and column;
  - matchings in the bipartite graph $K(n, n)$.

  It would be interesting to find combinatorial proofs for the conjectures above.

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