PARTIALLY ORDERED PATTERNS AND COMPOSITIONS

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Abstract

A partially ordered (generalized) pattern (POP) is a generalized pattern some of whose letters are incomparable, an extension of generalized permutation patterns introduced by Babson and Steingrímsson. POPs were introduced in the symmetric group by Kitaev [19, 21], and studied in the set of k-ary words by Kitaev and Mansour [22]. Moreover, Kitaev et al. [23] introduced segmented POPs in compositions. In this paper, we study avoidance of POPs in compositions and generalize results for avoidance of POPs in permutations and words. Specifically, we obtain results for the generating functions for the number of compositions that avoid shuffle patterns and multi-patterns. In addition, we give the generating function for the distribution of the maximum number of non-overlapping occurrences of a segmented POP τ (that is allowed to have repeated letters) among the compositions of n with m parts in a given set, provided we know the generating function for the number of compositions of n with m parts in the given set that avoid τ. This result is a q-analogue of the main result in [22].

Keywords: Compositions, partially ordered (generalized) patterns, non-overlapping occurrences, generating functions.

2000 Mathematics Subject Classification: 05A05, 05A15

1. Introduction

Pattern avoidance was originally studied in permutations (see [7, 26]), and the patterns studied were also permutation patterns. Generalizations in several directions took place: 1) Looking at pattern avoidance in permutations with different types of patterns and avoidance of sets of patterns (see [19] and references therein), and 2) asking the same questions for words (see [3, 5, 6, 22]). Independently, several authors (see [8, 9, 10, 11, 12, 13, 14, 15]) gave results on enumerating compositions of n with parts in a given set A according to rises, levels and drops (which can be considered as the simplest 2-letter patterns). Heubach and Mansour (see [16, 17]) combined these two areas by giving results on the generating function for the number of compositions of n with m parts in a set A that avoid

3-letter patterns. Moreover, Kitaev et al. [23] introduced segmented partially ordered (generalized) patterns in compositions.

In this paper we generalize some of the results in the literature on pattern avoidance in permutations [19], [21] and words [22] by studying pattern avoidance of partially ordered patterns (POPs) in compositions. Section 2 contains basic definitions and terminology. In Section 3, we give a general result that expresses the generating function of the number of compositions that avoid a POP composed of two smaller patterns in terms of the generating functions for the smaller patterns. We apply this result (Theorem 3.2) to two specific types of POPs, namely shuffle patterns and multi-patterns and show equivalence for families of patterns of each type. We close in Section 4 by giving a result for the maximum number of non-overlapping occurrences of a POP in a composition, which is a generalization of a theorem proved by Kitaev [19, Theorem 32] for permutations and by Kitaev and Mansour [22, Theorem 5.1] for words.

2. Definitions and Terminology

Let $\mathbb{N}$ be the set of all positive integers, and let A be any ordered finite set of positive integers, say $A = \{a_1, a_2, \ldots, a_k\}$, where $a_1 < a_2 < a_3 < \cdots < a_k$. (An "ordered set" in this paper will always refer to a set whose elements are listed in increasing order.) Also, let $[k]^n$ denote the set of all words of length n over the (totally ordered) alphabet $[k] = \{1, 2, \ldots, k\}$.

A *composition* $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of *summands* or *letters*, namely m, is called the number of *parts* of the composition. For any ordered set $A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{N}$, we denote the set of all compositions of n with parts in A (resp. with m parts in A) by C_n^A (resp. $C_{n;m}^A$).

A *generalized pattern* τ is a word in $[\ell]^m$ (possibly with dashes between some letters) that contains each letter from $[\ell]$ (possibly with repetitions). Generalized patterns that contain dashes in all possible positions (e.g., 2-1-4-3) are called *classical patterns*. Note that classical patterns place no adjacency requirements on occurrences of the letters of a pattern in words or compositions. If all the dashes are removed, we have a *consecutive*, or *segmented*, *pattern*. For ease of readability, we will refer to generalized patterns simply as patterns in the remainder of this paper.

We say that a composition $\sigma \in C_n^A$ *contains* a pattern τ if σ contains a subsequence isomorphic to τ in which the entries corresponding to consecutive entries of τ (those not separated by a dash) must be adjacent. Otherwise, we say that σ *avoids* τ and write $\sigma \in C_n^A(\tau)$. Thus, $C_n^A(\tau)$ denotes the set of all compositions of n with parts in A that avoid τ. Moreover, if T is a set of patterns, then $C_n^A(T)$ denotes the set of all compositions of n with parts in A that avoid each pattern from T simultaneously. For example, 241874 avoids 312 and contains three occurrences of 1-32, namely 287, 274 and 487. (Note that 284 is not an occurrence of 1-32 due to the adjacency requirement).

Kitaev [19], [21] introduced *partially ordered patterns* (*POPs*)[1] on permutations, which extend *generalized permutation patterns* introduced by Babson and Steingrímsson [2]. Specifically, a POP τ is a word consisting of letters from a partially ordered alphabet $\mathcal{T}$ such that the letters in τ constitute an order ideal in $\mathcal{T}$. If letters a and b are incomparable in a POP τ, then the relative size of the letters in σ corresponding to a and b is unimportant in an occurrence of τ in σ. For instance, if $\mathcal{T} = \{1, 1', 2'\}$ and the only relation is $1' < 2'$, then the sequence 31254 has two occurrences of $\tau = 11'2'$, namely

[1]In [19], POPs are called *POGPs* (*Partially Ordered Generalized Patterns*). We use POPs instead to shorten the notation in this paper.

312 and 125. As for generalized patterns, if a POP $\tau = \tau_1 \dots \tau_k$ has a dash between, say, τ_i and τ_{i+1}, then in an occurrence of τ in a composition σ, the letters corresponding to τ_i and τ_{i+1} do not have to be adjacent. For example, for $\mathcal{T}$ given above, if $\tau = 1\text{-}1'2'$, then the composition 113425 contains seven occurrences of τ, namely 113, 134 twice, 125 twice, 325, and 425.

Following [19] and [22], we consider two particular classes of POPs – shuffle patterns and multi-patterns – which allows us to give an analogue of the main results in [19] and [22] for compositions. Let $\{\tau_0, \tau_1, \dots, \tau_s\}$ be a set of consecutive patterns. A *multi-pattern* is of the form $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ and a *shuffle pattern* is of the form $\tau = \tau_0\text{-}a_1\text{-}\tau_1\text{-}a_2\text{-}\cdots\text{-}\tau_{s-1}\text{-}a_s\text{-}\tau_s$, where each letter of τ_i is incomparable with any letter of τ_j whenever $i \neq j$. In addition, the letters a_i are either all greater or all smaller than any letter of τ_j for any i and j. For example, $1'\text{-}2\text{-}1''$ is a shuffle pattern, and $1'\text{-}1''$ is a multi-pattern. Clearly, we can get a multi-pattern from a shuffle pattern by removing all the letters a_i. Furthermore, there is a connection between avoidance of a POP and multi-avoidance of generalized patterns in compositions. For example, avoiding the POP $2'\text{-}1\text{-}2''$ is the same as simultaneously avoiding the patterns 2-1-2, 3-1-2, and 2-1-3 (similar to [22, Proposition 2.7]).

3. POPs in compositions with parts in a given set

We will now derive results on avoidance of POPs in compositions. In order to distinguish which letters are comparable and which ones are not, we will use primes in the following way. If two letters, say 1 and 2, have the same number of primes, say two, then they are comparable and naturally $1'' < 2''$. Any two letters with a different number of primes are incomparable. Unless dealing with shuffle or multi-patterns, if a letter in a POP has no primes, then that letter is greater than every letter with one or more primes and we will emphasize this fact by using a value that is bigger than those for the primed letters. For example, in $\tau = 1'\text{-}2\text{-}1''$, the second letter is the greatest one and the first and the last letters are incomparable to each other. The composition $\sigma = 31421$ has five occurrences of τ, namely 342, 341, 142, 141, and 121.

Let $C_\tau^A(x) = \sum_{n\geq 0} |C_n^A(\tau)| x^n$ (resp. $C_\tau^A(x;m) = \sum_{n\geq 0} |C_{n;m}^A(\tau)| x^n$ and $C_\tau^A(x,y) = \sum_{n,m\geq 0} |C_{n;m}^A(\tau)| x^n y^m$) denote the generating function for the numbers $|C_n^A(\tau)|$ (resp. $|C_{n;m}^A(\tau)|$) of compositions in C_n^A (resp. $C_{n;m}^A$) avoiding the pattern τ. For example, if $A = \{a_1, a_2, \dots, a_k\}$ is any ordered set and $\tau = 1'\text{-}2\text{-}1''$, then we have

$$C_{1'\text{-}2\text{-}1''}^A(x,y) = \frac{1}{\prod_{a\in A}(1-x^a y)^2} - \sum_{a\in A} \frac{x^a y}{\prod_{a\leq b\in A}(1-x^b y)^2}. \tag{3.1}$$

This result follows from the specific structure of the compositions σ that avoid $\tau = 1'\text{-}2\text{-}1''$. If σ avoids τ, and σ contains $s > 0$ copies of the letter a_k, then the letters a_k can only appear as blocks on the left and right end of σ. If σ contains no a_k, then $\sigma \in C_{n,m}^{A'}(\tau)$ where $A' = A - \{a_k\}$. So, for all $n \geq 0$, we have

$$C_\tau^A(x;m) = \sum_{i=0}^{m-1} (i+1) x^{ia_k} C_\tau^{A'}(x;m-i) + x^{ma_k},$$

since the generating function for the possibilities to place i letters a_k into σ is given by $(i+1)C_\tau^{A'}(x;m-i)$, for $0 \leq i < m$, and by $x^{m\,a_k}$ for $i = m$. Thus, for $m \geq 2$,

$$C_\tau^A(x;m) - 2x^{a_k} C_\tau^A(x;m-1) = C_\tau^{A'}(x;m) - x^{2a_k}\left(\sum_{s=0}^{m-3} (s+1) x^{sa_k} C_\tau^{A'}(x;m-2-s) - x^{(m-2)a_k}\right)$$

or equivalently,

$$C_\tau^A(x;m) - 2x^{a_k}C_\tau^A(x;m-1) + x^{2a_k}C_\tau^A(x;m-2) = C_\tau^{A'}(x;m),$$

together with $C_\tau^A(x;0) = 1$ and $C_\tau^A(x;1) = \sum_{a\in A} x^a y$. Multiplying both sides of the recurrence above by y^m, summing over all $m \geq 2$ and using induction on elements of A together with the fact that $C_\tau^{\{a_1\}}(x,y) = \frac{1}{1-x^{a_1}y} = \frac{1}{(1-x^{a_1}y)^2} - \frac{x^{a_1}y}{(1-x^{a_1}y)^2}$, we get (3.1). Equation (3.1) for $x = 1$ and $A = [k]$ gives the corresponding result for words [22, Equation 2.1].

In order to prove general results, it is convenient to introduce the notion of quasi-avoidance. Let τ be a consecutive pattern. A composition σ *quasi-avoids* τ if σ has exactly one occurrence of τ and this occurrence consists of the $|\tau|$ rightmost parts of σ, where $|\tau|$ denotes the number of letters in τ. For example, the composition 4112234 quasi-avoids the pattern 1123, whereas the compositions 5223411 and 1123346 do not.

First, relate the generating function for the number of compositions avoiding a given pattern τ with the generating function for the number of compositions that quasi-avoid τ.

Lemma 3.1. *Let τ be a non-empty consecutive pattern. Let $D_\tau^A(x,y)$ denote the generating function for the number of compositions in $C_{n;m}^A$ that quasi-avoid τ. Then*

$$D_\tau^A(x,y) = 1 + C_\tau^A(x,y)\left(y\sum_{a\in A} x^a - 1\right). \tag{3.2}$$

Proof. We use arguments similar to in the proof of [19, Proposition 4]. Adding the part a to a composition with $m-1$ parts that avoids τ creates either a composition with m parts that still avoids τ or that quasi-avoids τ. Thus, for $m \geq 1$,

$$D_\tau^A(x;m) = \left(\sum_{a\in A} x^a\right) C_\tau^A(x;m-1) - C_\tau^A(x;m).$$

Multiplying both sides of this equality by y^m and summing over all natural numbers m we get the desired result. □

Lemma 3.1 for $A = [k]$ and $x = 1$ gives the corresponding result for words [22, Proposition 2.4].

We now obtain a general theorem that is a good auxiliary tool for calculating the generating function for the number of compositions that avoid a given POP.

Theorem 3.2. *Let $A = \{a_1, \ldots, a_k\}$ be any ordered finite set of positive integers. Suppose $\tau = \tau_0$-ϕ, where ϕ is an arbitrary POP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have*

$$C_\tau^A(x,y) = C_{\tau_0}^A(x,y) + C_\phi^A(x,y)D_{\tau_0}^A(x,y).$$

Proof. To find $C_\tau^A(x,y)$, we observe that there are two possibilities: either σ avoids τ_0, or σ does not avoid τ_0. In the first case, the generating function is given by $C_{\tau_0}^A(x,y)$. If σ does not avoid τ_0, then we can write σ in the form $\sigma = \sigma_1\sigma_2\sigma_3$, where $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0, and σ_2 is order isomorphic to τ_0. Clearly, σ_3 must avoid ϕ, thus, the generating function is equal to $C_\phi^A(x,y)D_{\tau_0}^A(x,y)$, and we obtain the stated result. □

Theorem 3.2 can be used for reduction, but also to easily compute the generating function for a new pattern from the generating function of a known pattern. We will use the notion of equivalence of patterns to obtain several results that hold for whole families of patterns. Two POPs τ and ϕ are said to be *equivalent*, and we write $\tau \equiv \phi$, if the number of compositions in $C^A_{n;m}$ that avoid τ is equal to the number of compositions in $C^A_{n;m}$ that avoid ϕ for all n, m.

Let $A = \{a_1, \dots, a_k\}$ be any ordered finite set of positive integers and $\sigma = \sigma_1\sigma_2 \dots \sigma_m \in C^A_{n;m}$. Then the *reverse* $R(\sigma)$ of a composition σ is the composition $\sigma_m \dots \sigma_2\sigma_1$. We call this bijection of $C^A_{n;m}$ to itself *trivial*. (The other trivial bijection is I, the identity bijection). Note that the complement operation defined for permutations and words is not defined for compositions. It is easy to see that $\tau \equiv R(\tau)$ for any pattern τ. For example, the number of compositions that avoid the pattern 21-2 is the same as the number of compositions that avoid the pattern 2-12.

In the following two subsections we obtain results for two specific classes of POPs – shuffle patterns and multi-patterns.

3.1. **Shuffle patterns in compositions.** We consider the shuffle patterns τ-ℓ-ν and τ-1-ν, where ℓ (resp. 1) is the greatest (resp. smallest) element of the pattern.

Theorem 3.3. *Let $A = \{a_1, a_2, \dots, a_k\}$ be any ordered set of positive integers.*

(1) *Let ϕ be the shuffle pattern τ-ℓ-ν. Then for all $k \geq \ell$,*

$$C^A_\phi(x,y) = \frac{C^{A-\{a_k\}}_\phi(x,y) - x^{a_k}yC^{A-\{a_k\}}_\tau(x,y)C^{A-\{a_k\}}_\nu(x,y)}{(1 - x^{a_k}yC^{A-\{a_k\}}_\tau(x,y))(1 - x^{a_k}yC^{A-\{a_k\}}_\nu(x,y))}.$$

(2) *Let ψ be the shuffle pattern τ-1-ν. Then for all $k \geq \ell$,*

$$C^A_\psi(x,y) = \frac{C^{A-\{a_1\}}_\psi(x,y) - x^{a_1}yC^{A-\{a_1\}}_\tau(x,y)C^{A-\{a_1\}}_\nu(x,y)}{(1 - x^{a_1}yC^{A-\{a_1\}}_\tau(x,y))(1 - x^{a_1}yC^{A-\{a_1\}}_\nu(x,y))}.$$

Proof. We derive a recurrence relation for $C^A_\phi(x,y)$ where $\phi = \tau$-ℓ-ν. Let $\sigma \in C^A_{n,m}(\phi)$ be such that it contains exactly s copies of the letter a_k. If $s = 0$, then the generating function for the number of such compositions is $C^{A'}_\phi(x,y)$, where $A' = A - \{a_k\}$. For $s \geq 1$, we write $\sigma = \sigma_0 a_k \sigma_1 a_k \cdots a_k \sigma_s$, where σ_j is a ϕ-avoiding composition with parts in A', for $j = 0, 1, \dots, s$. Then either σ_j avoids τ for all j, or there exists a j_0 such that σ_{j_0} contains τ, σ_j avoids τ for all $j = 0, 1, \dots, j_0 - 1$ and σ_j avoids ν for any $j = j_0 + 1, \dots, s$. In the first case, the generating function for the number of such compositions is $x^{sa_k}y^s\left(C^{A'}_\tau(x,y)\right)^{s+1}$. In the second case, the generating function is given by

$$x^{sa_k}y^s \sum_{j=0}^{s} \left(C^{A'}_\tau(x,y)\right)^j \left(C^{A'}_\nu(x,y)\right)^{s-j} (C^{A'}_\phi(x,y) - C^{A'}_\tau(x,y)).$$

Therefore, we get

$$\begin{aligned} C^A_\phi(x,y) = C^{A'}_\phi(x,y) + C^{A'}_\phi(x,y) \sum_{s\geq 1} x^{sa_k}y^s \sum_{j=0}^{s} \left(C^{A'}_\tau(x,y)\right)^j \left(C^{A'}_\nu(x,y)\right)^{s-j} \\ - \sum_{s\geq 1} x^{sa_k}y^s \sum_{j=1}^{s} \left(C^{A'}_\tau(x,y)\right)^j \left(C^{A'}_\nu(x,y)\right)^{s+1-j}, \end{aligned}$$

or equivalently,

$$C_\phi^A(x,y) = (C_\phi^{A'}(x,y) - x^{a_k} y C_\tau^{A'}(x,y) C_\nu^{A'}(x,y)) \sum_{s\geq 0} x^{sa_k} y^s \sum_{j=0}^{s} \left(C_\tau^{A'}(x,y)\right)^j \left(C_\nu^{A'}(x,y)\right)^{s-j}.$$

Hence, using the identity $\sum_{n\geq 0} x^n \sum_{j=0}^{n} p^j q^{n-j} = \frac{1}{(1-xp)(1-xq)}$ we get the desired result (1). Using similar arguments and replacing a_1 by a_k, we obtain (2). □

For certain shuffle patterns ϕ we can compute the generating function $C_\phi^A(x,y)$ explicitly, using the recursion given in Theorem 3.3.

Example 3.4. *Let $A = \{a_1, \ldots, a_k\}$ be any ordered set of positive integers and $\phi = 1'\text{-}2\text{-}1''$ (resp. $\psi = 2'\text{-}1\text{-}2''$). Here $\tau = \nu = 1$, so $C_\tau^A(x,y) = C_\nu^A(x,y) = 1$ for any A, since only the empty composition avoids τ. Hence,*

$$C_\phi^A(x,y) = \frac{1}{(1-x^{a_k}y)^2}(C_\phi^{A-\{a_k\}}(x,y) - x^{a_k}y).$$

Also, $C_\phi^{\{a_1\}}(x,y) = \frac{1}{1-x^{a_1}y}$ as for any m, only the composition $\underbrace{a_1a_1\ldots a_1}_{m \text{ times}}$ avoids ϕ and therefore,

$$C_{1'\text{-}2\text{-}1''}^A(x,y) = \frac{1}{\prod_{a\in A}(1-x^a y)^2} - \sum_{a\in A} \frac{x^a y}{\prod_{a\leq b\in A}(1-x^b y)^2},$$

the result obtained in Equation (3.1) directly. Likewise, we obtain

$$C_{2'\text{-}1\text{-}2''}^A(x,y) = \frac{1}{\prod_{a\in A}(1-x^a y)^2} - \sum_{a\in A} \frac{x^a y}{\prod_{a\geq b\in A}(1-x^b y)^2}.$$

We now give two corollaries to Theorem 3.3.

Corollary 3.5. *Let $\phi = \tau\text{-}\ell\text{-}\nu$ (resp. $\phi = \tau\text{-}1\text{-}\nu$) be a shuffle pattern, and let $f(\phi) = f_1(\tau)\text{-}\ell\text{-}f_2(\nu)$ (resp. $f(\phi) = f_1(\tau)\text{-}1\text{-}f_2(\nu)$), where $f_1, f_2 \in \{R, I\}$ are any trivial bijections. Then $\phi \equiv f(\phi)$.*

Proof. Using Theorem 3.3, and the fact that the number of compositions in $C_{n;m}^A$ avoiding τ (resp. ν) and $f_1(\tau)$ (resp. $f_2(\nu)$) have the same generating functions, we get the desired result. □

Corollary 3.6. *For any shuffle pattern $\tau\text{-}\ell\text{-}\nu$ (resp. $\tau\text{-}1\text{-}\nu$), we have $\tau\text{-}\ell\text{-}\nu \equiv \nu\text{-}\ell\text{-}\tau$ (resp. $\tau\text{-}1\text{-}\nu \equiv \nu\text{-}1\text{-}\tau$).*

Proof. Corollary 3.5 yields that the pattern $\tau\text{-}\ell\text{-}\nu$ (resp. $\tau\text{-}1\text{-}\nu$) is equivalent to the pattern $\tau\text{-}\ell\text{-}R(\nu)$ (resp. $\tau\text{-}1\text{-}R(\nu)$), which is equivalent to the pattern $R(\tau\text{-}\ell\text{-}R(\nu)) = \nu\text{-}\ell\text{-}R(\tau)$ (resp. $R(\tau\text{-}1\text{-}R(\nu)) = \nu\text{-}1\text{-}R(\tau)$). Finally, we use Corollary 3.5 one more time to get the desired result. □

3.2. **Multi-patterns in compositions.** We now look at the second class of patterns. Recall that a multi-pattern is of the form $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$, where $\{\tau_1, \ldots, \tau_s\}$ is a set of consecutive patterns and each letter of τ_i is incomparable with any letter of τ_j whenever $i \neq j$.

The simplest non-trivial example of a multi-pattern is the pattern $\phi = 1\text{-}1'2'$. To avoid ϕ is the same as to avoid the patterns 1-12, 1-23, 2-12, 2-13, and 3-12 simultaneously. To count the number of compositions in $C^A_{n;m}(1\text{-}1'2')$, we choose the leftmost letter of $\sigma \in C^A_{n;m}(1\text{-}1'2')$ in k ways, namely $a_1, \ldots, a_k$, and observe that all the other letters of σ must be in non-increasing order. Hence,

$$C^A_{1\text{-}1'2'}(x, y) = 1 + \frac{y \sum_{a \in A} x^a}{\prod_{a \in A}(1 - x^a y)}.$$

More generally, using Lemma 3.1 and Theorem 3.2, we get the following theorem that is the basis for calculating the number of compositions that avoid a multi-pattern, and therefore is the main result for multi-patterns in this paper.

Theorem 3.7. *Let $A = \{a_1, \ldots, a_k\}$ be any ordered finite set of positive integers and let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern. Then*

$$C^A_\tau(x, y) = \sum_{j=1}^{s} C^A_{\tau_j}(x, y) \prod_{i=1}^{j-1} \left[\left(y \sum_{a \in A} x^a - 1 \right) C^A_{\tau_i}(x, y) + 1 \right].$$

Example 3.8. *Let $A = \{a_1, \ldots, a_k\}$ be any ordered set of positive integers. Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern such that τ_j is equal to either 12 or 21, for $j = 1, 2, \ldots, s$. It is easy to see that $C^A_{12}(x, y) = C^A_{21}(x, y) = \frac{1}{\prod_{a \in A}(1 - x^a y)}$ and we obtain from Theorem 3.7*

$$C^A_\tau(x, y) = \frac{1 - \left(1 + \frac{y \sum_{a \in A} x^a - 1}{\prod_{a \in A}(1 - x^a y)}\right)^s}{1 - y \sum_{a \in A} x^a}.$$

Using arguments similar to those in the proof of [22, Theorem 4.1] we get the following theorem which is an analogue to [19, Theorem 21] and [22, Theorem 4.1].

Theorem 3.9. *Let $\tau = \tau_0\text{-}\tau_1$ and $\phi = f_1(\tau_0)\text{-}f_2(\tau_1)$, where f_1 and f_2 are any of the trivial bijections. Then $\tau \equiv \phi$.*

Proof. First we prove that $\tau = \tau_0\text{-}\tau_1 \equiv \tau_0\text{-}f(\tau_1)$, where f is a trivial bijection. Suppose that $\sigma = \sigma_1\sigma_2\sigma_3$ avoids τ and $\sigma_1\sigma_2$ has exactly one occurrence of τ_0, namely σ_2. Then σ_3 must avoid τ_1, so $f(\sigma_3)$ avoids $f(\tau_1)$ and $\sigma_f = \sigma_1\sigma_2 f(\sigma_3)$ avoids ϕ. The converse is also true, if σ_f avoids ϕ then σ avoids τ. Since any composition either avoids τ_0 or can be factored as above, we have a bijection between the class of compositions avoiding τ and the class of compositions avoiding ϕ. Thus $\tau_0\text{-}\tau_1 \equiv \tau_0\text{-}f(\tau_1)$. Using this result as well as the properties of trivial bijections we get

$$\begin{aligned} \tau \equiv \tau_0\text{-}f_2(\tau_1) \equiv R(\tau_0\text{-}f_2(\tau_1)) \equiv R(f_2(\tau_1))\text{-}R(\tau_0) \equiv \\ \equiv R(f_2(\tau_1))\text{-}f_1(R(\tau_0)) \equiv R(f_2(\tau_1))\text{-}R(f_1(\tau_0)) \equiv f_1(\tau_0)\text{-}f_2(\tau_1). \end{aligned}$$

□

Corollary 3.10. *The multi-patterns $\tau_1\text{-}\tau_2$ and $\tau_2\text{-}\tau_1$ are equivalent.*

Proof. From Theorem 3.9, using the properties of the trivial bijection R, we get

$$\tau_1\text{-}\tau_2 \equiv \tau_1\text{-}R(\tau_2) \equiv R(R(\tau_2))\text{-}R(\tau_1) \equiv \tau_2\text{-}R(R(\tau_1)) \equiv \tau_2\text{-}\tau_1.$$

□

We can obtain an even more general result.

Theorem 3.11. *Suppose we have multi-patterns $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ and $\phi = \phi_1\text{-}\phi_2\text{-}\cdots\text{-}\phi_s$, where $\tau_1\tau_2\dots\tau_s$ is a permutation of $\phi_1\phi_2\dots\phi_s$. Then $\tau \equiv \phi$.*

Proof. We use induction on k. For $s = 2$, the statement follows from Corollary 3.10. Suppose the statement is true for all $k < s$. If the composition σ has no occurrences of ϕ_1, then it obviously avoids both τ and ϕ. Otherwise, we can write $\sigma = \sigma_1\sigma_2\sigma_3$, where $\sigma_1\sigma_2$ quasi-avoids ϕ_1. Then σ_3 has to avoid $\phi_2\text{-}\cdots\text{-}\phi_s$. Since the ϕ_i are incomparable, it is irrelevant from which letters $\sigma_1\sigma_2$ is built, and we can apply the inductive hypothesis to $\phi_2\text{-}\cdots\text{-}\phi_s$. We can rearrange $\phi'_2, \dots \phi'_k$ of $\phi_2, \dots \phi_k$ in such a way that the blocks in $\tau_1\tau_2\dots\tau_s$ corresponding to $\phi_2, \dots, \phi_s$ are arranged in the same order as the τ's. Then

$$\phi = \phi_1\text{-}\phi_2\text{-}\cdots\text{-}\phi_s \equiv \phi_1\text{-}\phi'_2\text{-}\cdots\text{-}\phi'_s \equiv R(\phi'_s)\text{-}\cdots\text{-}R(\phi'_2)\text{-}R(\phi_1). \tag{3.3}$$

Now we consider two cases: Either $\tau_k \neq \phi_1$ or $\tau_k = \phi_1$. In the first case, we apply the hypothesis to the pattern $R(\phi'_s)\text{-}\cdots\text{-}R(\phi'_2)\text{-}R(\phi_1)$, with the role of ϕ_1 played by $R(\phi'_s)$. Thus, we can move the pattern $R(\phi_1)$ to the correct place somewhere to the left of $R(\phi'_2)$, then apply the bijection R to obtain that $\tau \equiv \phi$. In the second case, we obtain

$$\phi \equiv R(\phi'_s)\text{-}\cdots\text{-}R(\phi'_2)\text{-}R(\phi_1) \equiv R(\phi'_s)\text{-}\cdots\text{-}R(\phi_1)\text{-}R(\phi'_2) \equiv \phi'_2\text{-}\phi_1\text{-}\cdots\text{-}\phi'_s \equiv \phi'_2\text{-}\phi'_s\text{-}\cdots\text{-}\phi_1 = \tau.$$

The first equivalence follows from (3.3); the second one follows from the inductive hypothesis. Applying the bijection R together with $R(R(x)) = x$ and the inductive hypothesis once more gives the remaining equivalences. □

4. Non-overlapping occurrences of POPs in compositions

Kitaev [19] and Mendes and Remmel [24, 25] proved the following result on the distribution of non-overlapping patterns in permutations: Let τ-nlap(σ) be the maximum number of non-overlapping occurrences of a consecutive pattern τ in a permutation σ where two occurrences of τ are said to overlap if they contain any of the same integers. Then

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{\sigma \in S_n} y^{\tau\text{-nlap}(\sigma)} = \frac{A(x)}{(1-y) + y(1-x)A(x)}, \tag{4.1}$$

where $A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} |\sigma \in S_n \ : \ \sigma \text{ avoids } \tau|$. In other words, if the exponential generating function for the number of permutations in S_n avoiding τ is known, then so is the bivariate generating function for the entire distribution of τ-nlap. Kitaev and Mansour [22, Theorem 5.1] found an analogue to (4.1) in case of words. We now prove a corresponding result for compositions.

Let τ be an arbitrary consecutive pattern. We say that two patterns *overlap* in a composition if they contain any of the same letters of the composition. Using Theorem 3.7 for the multi-pattern $\tau\text{-}\tau\text{-}\cdots\text{-}\tau$ allows us to obtain the generating function for the entire distribution of the maximum number of non-overlapping occurrences of a pattern τ in compositions.

The simplest consecutive pattern is a descent (or drop) in a composition, which occurs at position i if $\sigma_i > \sigma_{i+1}$. Clearly, two descents i and j overlap if $j = i+1$. In particular, we can define the statistic *maximum number of non-overlapping descents*, or MND, in a composition. For example, MND(333211) = 1 whereas MND(13321111432111) = 3 (namely 32, 43 and 21). Obviously, this statistic, maximum number of non-overlapping patterns, can be defined for any consecutive pattern τ, and we obtain the following result.

Theorem 4.1. *Let A be any ordered set of positive integers and let τ be a consecutive pattern. Then*

$$\sum_{n,m\geq 0}\sum_{\sigma\in C^A_{n;m}} t^{\tau\text{-}nlap(\sigma)}x^n y^m = \frac{C^A_\tau(x,y)}{1-t\left[\left(y\sum_{a\in A}x^a-1\right)C^A_\tau(x,y)+1\right]},$$

where τ-nlap(σ) is the maximum number of non-overlapping occurrences of τ in σ.

Proof. We fix a natural number s and consider the multi-pattern $\Phi_s = \tau\text{-}\tau\text{-}\cdots\text{-}\tau$ with s copies of τ. If a composition avoids Φ_s then it has at most $s-1$ non-overlapping occurrences of τ. Theorem 3.7 yields

$$C^A_{\Phi_s}(x,y) = \sum_{j=1}^{s} C^A_\tau(x,y)\prod_{i=1}^{j-1}\left[\left(y\sum_{a\in A}x^a-1\right)C^A_\tau(x,y)+1\right].$$

Therefore, the generating function for the number of compositions that have exactly s non-overlapping occurrences of the pattern τ is given by

$$C^A_{\Phi_{s+1}}(x,y) - C^A_{\Phi_s}(x,y) = C^A_\tau(x,y)\left[\left(y\sum_{a\in A}x^a-1\right)C^A_\tau(x,y)+1\right]^s.$$

Hence,

$$\sum_{n,m\geq 0}\sum_{\sigma\in C^A_{n;m}} t^{\tau\text{-nlap}(\sigma)}x^n y^m = \sum_{s\geq 0} t^s C^A_\tau(x,y)\left[\left(y\sum_{a\in A}x^a-1\right)C^A_\tau(x,y)+1\right]^s,$$

or, equivalently,

$$\sum_{n,m\geq 0}\sum_{\sigma\in C^A_{n;m}} t^{\tau\text{-nlap}(\sigma)}x^n y^m = \frac{C^A_\tau(x,y)}{1-t\left[\left(y\sum_{a\in A}x^a-1\right)C^A_\tau(x,y)+1\right]}.$$

□

Note that Theorem 4.1 is a q-analogue to [22, Theorem 5.1], which is the main result of [22] (set $x = 1$ to get the result for words). We use Theorem 4.1 to obtain the distribution for *MND*, the maximum number of non-overlapping descents.

Example 4.2. *Let A be any ordered set of positive integers. If we consider descents (the pattern 12) then $C^A_{12}(x,y) = \frac{1}{\prod_{a\in A}(1-x^a y)}$, hence the distribution of MND is given by the formula:*

$$\sum_{n,m\geq 0}\sum_{\sigma\in C^A_{n;m}} t^{12\text{-}nlap(\sigma)}x^n y^m = \frac{1}{\prod_{a\in A}(1-x^a y)+t\left(1-y\sum_{a\in A}x^a-\prod_{a\in A}(1-x^a y)\right)}.$$

Specifically, the distribution of MND on the set of compositions of n with parts in $A = \{1, 2\}$ is given by

$$\frac{1}{(1-x)(1-x^2)-x^3t} = \sum_{s\geq 0} \frac{x^{3s}}{(1-x)^{2s+2}(1+x)^{s+1}} t^s.$$

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