Equidistribution of mesh patterns of short length on alternating permutations

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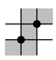
Abstract. The distribution of mesh patterns of length 2 over all permutations has recently been completely classified into 105 equivalence classes through a series of papers by several authors. In this paper we initiate the corresponding classification over the class of alternating permutations. The alternating setting is substantially richer: we prove that the 1024 mesh patterns of length 2 fall into exactly 203 distribution-equivalence classes on alternating permutations. Several equivalences over all permutations no longer hold after restricting to alternating permutations, while some new equivalences appear only in the alternating setting; moreover, some known all-permutation bijections do not preserve alternation. We overcome these obstacles using symmetry, a redundant-corner lemma, local toggles, block decompositions, an inclusion-exclusion argument, and modified bijections inspired by work of Han and Zeng.

Keywords: mesh pattern, alternating permutation, equidistribution, bijection.

AMS Subject Classification: 05A05, 05A15.

1 Introduction

Denote by S_n the set of permutations of $[n] = \{1, 2, \ldots, n\}$. A *mesh pattern* of length k is a pair (τ, R), where $\tau \in S_k$ and $R \subseteq [\![0, k]\!] \times [\![0, k]\!]$, where $[\![0, k]\!]$ denotes the interval of the integers from 0 to k. Let (i, j) denote the box with corners $(i, j), (i, j + 1), (i + 1, j + 1), (i + 1, j)$. For example,



is a mesh pattern of length 2 with $\tau = 12$ and $R = \{(0, 0), (1, 0), (1, 1), (1, 2), (2, 2)\}$. An occurrence of (τ, R) in $\pi \in S_n$ is an occurrence of τ whose associated shaded boxes contain no other points of the diagram of π.

The concept of mesh patterns was introduced by Brändén and Claesson [3]. The systematic study of avoidance for mesh patterns of length 2 began with Hilmarsson et al. [11]. Kitaev and Zhang [12] initiated the systematic study of their distributions. Subsequent progress in [9, 13, 17, 8, 19] led to the complete all-permutation classification into 105 distribution-equivalence classes. Other distributional questions for longer mesh patterns, especially joint equidistributions for length 3 patterns, have also been studied in [14, 15, 16, 7].

This paper extends the classification program for mesh patterns of length 2 to *alternating permutations*. We say that a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ is *up-down* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \pi_5 < \cdots$, and *down-up* if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \pi_5 > \cdots$.

Following [10], we use the term alternating permutation to refer to a permutation of either type. Let UD_n and DU_n denote, respectively, the sets of up-down and down-up permutations in S_n, and set

$$\mathcal{A}_n := UD_n \cup DU_n.$$

It is well-known that

$$|UD_n| = |DU_n| = E_n,$$

where E_n denotes the nth Euler number, with $E_0 = 1$. Alternating permutations are classical objects dating back to André [1, 2], whose secant-tangent enumeration is one of the earliest landmarks in enumerative combinatorics. They have since appeared in many forms, including binary increasing trees [5], permutation statistics [6, 18], run polynomials [4], and, more recently, in the study of maxima and minima statistics on alternating permutations [10], which is essentially a study of distributions of singleton mesh patterns (i.e., mesh patterns of length 1). For instance, it follows from [10] that the exponential generating function for the distribution of the singleton mesh pattern  on up-down permutations of even length is $(\sec t)^q$, where t records the length and q records the number of occurrences of the pattern; here, these occurrences are precisely the elements that are right-to-left maxima, as defined in Section 3.

For a mesh pattern p, let $p(\pi)$ denote the number of occurrences of p in a permutation π. Two mesh patterns p_1 and p_2 are *equidistributed* on alternating permutations if

$$\sum_{\pi \in \mathcal{A}_n} q^{p_1(\pi)} = \sum_{\pi \in \mathcal{A}_n} q^{p_2(\pi)}$$

for every $n \geq 0$, which we denote by $p_1 \sim p_2$. Our computations, performed for all 1024 patterns (τ, R) with $\tau \in \{12, 21\}$ and $R \subseteq [\![0,2]\!] \times [\![0,2]\!]$, produce 203 stable classes (i.e., classes that no longer split for larger values of n) through $n \leq 10$. The proofs below establish all 203 classes, yielding the following main result.

Theorem 1.1. *There are exactly* 203 *distribution-equivalence classes of mesh patterns of length* 2 *on* $\mathcal{A}_n$*.*

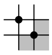
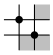
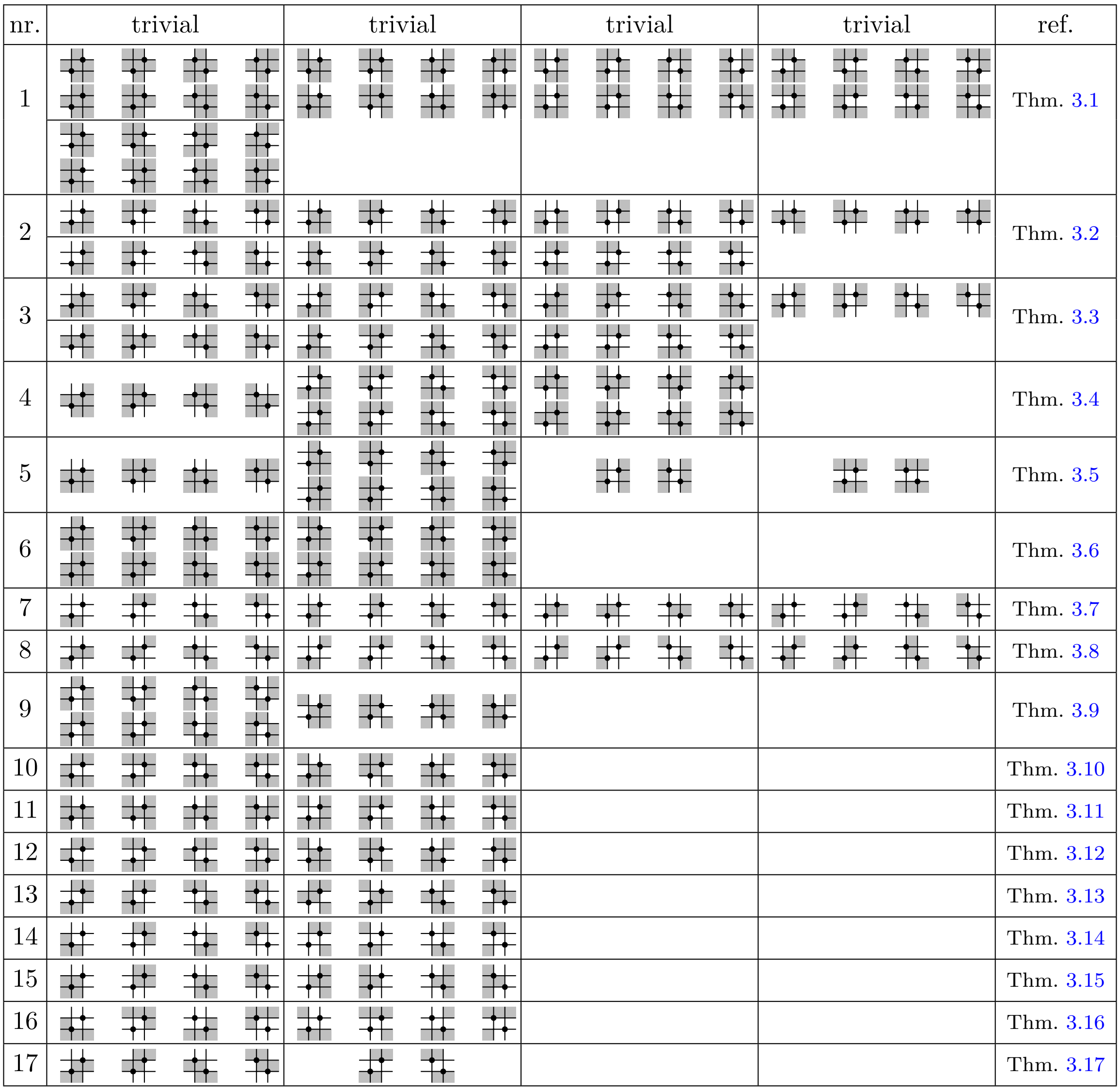
Interestingly, several equivalences that hold for all permutations in [17] no longer hold when restricted to alternating permutations (for example, for the patterns  and ), while some new equivalences arise only in the alternating setting (for example, for the patterns  and ). Moreover, some known bijections for all permutations do not preserve alternation; for example, the involution Ψ of Han and Zeng [9, Theorem 1.9], which proves  $\sim$  on the set of all permutations, does not preserve alternation.

Table 1 displays all patterns in the 17 classes that require nontrivial justification, with the trivial reverse/complement orbits joined into proof blocks by the redundant-corner lemma (Lemma 2.1). Table 2 displays all patterns in the remaining 186 classes, ordered by decreasing class size. Thus, the two tables together give all 203 computational distribution-equivalence classes.



nr.	trivial	trivial	trivial	trivial	ref.
1					Thm. 3.1
2					Thm. 3.2
3					Thm. 3.3
4					Thm. 3.4
5					Thm. 3.5
6					Thm. 3.6
7					Thm. 3.7
8					Thm. 3.8
9					Thm. 3.9
10					Thm. 3.10
11					Thm. 3.11
12					Thm. 3.12
13					Thm. 3.13
14					Thm. 3.14
15					Thm. 3.15
16					Thm. 3.16
17					Thm. 3.17

Table 1: Equivalence classes requiring class-specific arguments.

nr.	trivial	nr.	trivial
18		19	
20		21	
22		23	
24		25	
26		27	
28		29	
30		31	
32		33	
34		35	
36		37	
38		39	
40		41	
42		43	
44		45	
46		47	
48		49	
50		51	
52		53	
54		55	
56		57	
58		59	
60		61	
62		63	
64		65	
66		67	
68		69	
70		71	
72		73	
74		75	
76		77	
78		79	
80		81	
82		83	
84		85	
86		87	
88		89	
90		91	
92		93	
94		95	
96		97	
98		99	
100		101	

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Table 2 (continued)

nr.	trivial	nr.	trivial
102		103	
104		105	
106		107	
108		109	
110		111	
112		113	
114		115	
116		117	
118		119	
120		121	
122		123	
124		125	
126		127	
128		129	
130		131	
132		133	
134		135	
136		137	
138		139	
140		141	
142		143	
144		145	
146		147	
148		149	
150		151	
152		153	
154		155	
156		157	
158		159	
160		161	
162		163	
164		165	
166		167	
168		169	
170		171	
172		173	
174		175	
176		177	
178		179	
180		181	
182		183	

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Table 2 (continued)

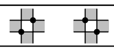
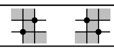
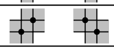
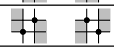
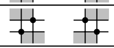
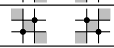
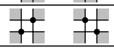
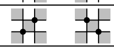
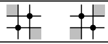
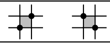
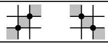
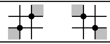
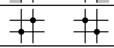
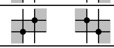
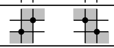
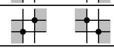
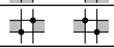
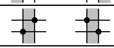
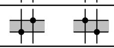
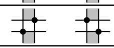
nr.	trivial	nr.	trivial
184		185	
186		187	
188		189	
190		191	
192		193	
194		195	
196		197	
198		199	
200		201	
202		203	

Table 2: Trivially equivalent classes.

The rest of the paper is organized as follows. In Section 2, we introduce two important lemmas. The first allows us to establish the equivalence of many pairs of patterns immediately, while the second is used several times throughout the paper in bijective arguments. Section 3 discusses all patterns requiring nontrivial justification, often via bijective arguments, thereby completing our classification of distribution-equivalence classes. Finally, in Section 4, we provide concluding remarks.

2 Preliminaries

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, its *reverse* is the permutation $\pi^r = \pi_n\pi_{n-1}\cdots\pi_1$ and its *complement* is the permutation $\pi^c = c(\pi_1)c(\pi_2)\cdots c(\pi_n)$ where $c(x) = n+1-x$. For example, for $\pi = 41325$, $\pi^r = 52314$, and $\pi^c = 25341$. It is easy to see that both operations are bijections on $\mathcal{A}_n$. However, we point out that the *inverse* (usual group-theoretic) operation does not preserve alternating property. As an example, the inverse of $\pi = 1423$ is $inv(\pi) = 1342$, which is not an alternating permutation.

We will repeatedly use the following *redundant-corner lemma* (*RCL*).

Lemma 2.1 (Redundant-corner lemma)**.** *Let* $R' = R \cup \{(0,0)\}$*, where* $(0,0) \notin R$*. If each of* $(0,1),(0,2),(1,0)$ *belongs to* R*, then* $(12,R)$ *and* $(12,R')$ *have the same number of occurrences in every alternating permutation and hence* $(12,R) \sim (12,R')$*. Applying the reverse, complement, or both gives the corresponding statements for the other three corners (namely,* $(0,2)$*,* $(2,0)$*, and* $(2,2)$*).*

Proof. Since $R \subset R'$, every $(12,R')$-occurrence is a $(12,R)$-occurrence. Conversely, let $a = \pi_i < b = \pi_j$ be a $(12,R)$-occurrence for $i < j$. Suppose that the box $(0,0)$ is nonempty. Then $i > 1$, and the shading of $(0,1)$ and $(0,2)$ gives $\pi_{i-1} < a$. Alternation therefore gives $\pi_{i+1} < a$. If $j = i+1$, then $b = \pi_{i+1} < a$, a contradiction. If $j > i+1$, then π_{i+1} lies in the shaded box $(1,0)$, again a contradiction. Hence $(0,0)$ is empty, so the occurrence is also a $(12,R')$-occurrence. □

Lemma 2.2 (Consecutive-value toggle lemma)**.** *For $\pi \in \mathcal{A}_n$, let $\mathrm{pos}_\pi(x)$ be the position of the element x. Fix $1 \le a < t \le n$ and set*

$$\mathcal{E}^{\mathrm{pre}}_{a,t} = \{\pi \in \mathcal{A}_n : \pi_1 = a \textit{ and } \mathrm{pos}_\pi(t) < \mathrm{pos}_\pi(s) \textit{ for every } s < a\},$$

and

$$\mathcal{E}^{\mathrm{post}}_{a,t} = \{\pi \in \mathcal{A}_n : \pi_1 = a \textit{ and } \mathrm{pos}_\pi(t) > \mathrm{pos}_\pi(s) \textit{ for every } s < a\}.$$

Both $|\mathcal{E}^{\mathrm{pre}}_{a,t}|$ and $|\mathcal{E}^{\mathrm{post}}_{a,t}|$ are independent of $t \in \{a+1, \ldots, n\}$. Moreover, the number of permutations in $\mathcal{A}_n$ ending in $1t$ is independent of $t \in \{2, \ldots, n\}$.

Proof. For $1 \le k < n$, define θ_k by interchanging the values k and $k+1$ when they are not adjacent in π, and leaving π unchanged when they are adjacent. Since no value lies strictly between k and $k+1$, the interchange preserves every adjacent inequality; hence θ_k is an involution on $\mathcal{A}_n$.

If $a + 1 < t \le n$, then θ_{t-1} fixes the first element a and maps $\mathcal{E}^{\mathrm{pre}}_{a,t}$ onto $\mathcal{E}^{\mathrm{pre}}_{a,t-1}$, and likewise maps $\mathcal{E}^{\mathrm{post}}_{a,t}$ onto $\mathcal{E}^{\mathrm{post}}_{a,t-1}$. Indeed, if $t-1$ and t are interchanged, then $t-1$ takes the former position of t. If they are adjacent and hence unchanged, they lie on the same side of every value smaller than a. Since θ_{t-1} is an involution on $\mathcal{A}_n$, its restriction to $\mathcal{E}^{\mathrm{pre}}_{a,t}$ (resp., $\mathcal{E}^{\mathrm{post}}_{a,t}$) is injective. Moreover, for any $\pi \in \mathcal{E}^{\mathrm{pre}}_{a,t-1}$ (resp., $\mathcal{E}^{\mathrm{post}}_{a,t-1}$), we have $\theta_{t-1}(\pi) \in \mathcal{E}^{\mathrm{pre}}_{a,t}$ (resp., $\mathcal{E}^{\mathrm{post}}_{a,t}$) and $\theta_{t-1}\big(\theta_{t-1}(\pi)\big) = \pi$. Hence θ_{t-1} gives a bijection in both cases. Thus successively applying $\theta_{t-1}, \theta_{t-2}, \ldots, \theta_{a+1}$ gives bijections from $\mathcal{E}^{\mathrm{pre}}_{a,t}$ to $\mathcal{E}^{\mathrm{pre}}_{a,a+1}$ and from $\mathcal{E}^{\mathrm{post}}_{a,t}$ to $\mathcal{E}^{\mathrm{post}}_{a,a+1}$.

For $2 \le t \le n$, let $\mathcal{B}_t = \{\pi \in \mathcal{A}_n : \pi_{n-1}\pi_n = 1t\}$. Suppose $t > 2$ and $\pi \in \mathcal{B}_t$. Since the final element t is adjacent only to the preceding element 1, and $t - 1 \ne 1$, the values $t-1$ and t are not adjacent in π. Thus θ_{t-1} interchanges them and fixes the penultimate element 1. Its image therefore ends in $1(t-1)$. Similarly, since θ_{t-1} is an involution on $\mathcal{A}_n$, its restriction is a bijection $\mathcal{B}_t \longrightarrow \mathcal{B}_{t-1}$. Consequently, $|\mathcal{B}_t| = |\mathcal{B}_{t-1}| = \cdots = |\mathcal{B}_2|$, which proves the last assertion. □

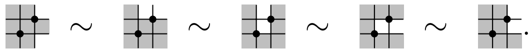
We call two patterns *trivially equivalent* if they are related by reverse, complement, their composition, or repeated applications of Lemma 2.1. Using only reverse and complement gives 272 preliminary classes; allowing applications of RCL reduces this to 240. The remaining nontrivial equivalences are handled below.

3 The nontrivial classes

Within each small block of Table 1, the patterns are equidistributed by reverse, complement, and Lemma 2.1. Hence the proofs below need only compare one representative from each block. As this paper only considers patterns of length two, the cases $n < 2$ are trivial. Hence, throughout the proofs below, it suffices to consider $n \ge 2$.

Theorem 3.1. *All patterns in Class 1 are equidistributed on $\mathcal{A}_n$.*

Proof. By the preceding observation, it is enough to prove



Each of these patterns has at most one occurrence. In the first pattern, the two fully shaded left columns force the selected positions to be 1 and 2, and the shaded rightmost box in the middle row forces the selected values to be consecutive. Thus it occurs exactly when $\pi_2 = \pi_1 + 1$.

Each of the other four patterns occurs exactly when $\pi_1 = 1$: their shaded rows or columns force the first selected value to be 1 and its position to be the first position; the remaining shading merely selects, respectively, the value 2, the last element, the value n, or the second element as its unique partner.

It remains to compare these two events. Deleting the initial 1 and taking the *reduced form* (that is, relabelling the i-th smallest element as i; this operation is often called *standardization* in this paper) gives a bijection

$$\{\pi \in \mathcal{A}_n : \pi_1 = 1\} \longrightarrow DU_{n-1}.$$

Similarly, deleting π_1 and taking the reduced form gives a bijection

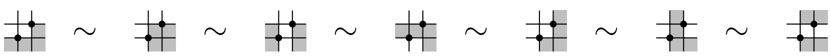
$$\{\pi \in \mathcal{A}_n : \pi_2 = \pi_1 + 1\} \longrightarrow DU_{n-1}.$$

Indeed, if $\sigma \in DU_{n-1}$ and $a = \sigma_1$, increment every element of σ that is at least a and then prepend a; this is the inverse of the second map. Hence both events have size $|DU_{n-1}|$. Since all five occurrence statistics take only the values 0 and 1, they have the same distribution. $\square$

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, an element π_i is a *right-to-left maximum* (resp., *right-to-left minimum*) if $\pi_i > \pi_j$ (resp., $\pi_i < \pi_j$) for every $j > i$. We denote the numbers of these elements by $\mathrm{rmax}(\pi)$ and $\mathrm{rmin}(\pi)$, respectively. *Left-to-right maximum* and *left-to-right minimum* are defined analogously, by comparing π_i with every π_j for $j < i$; their numbers are denoted by $\mathrm{lmax}(\pi)$ and $\mathrm{lmin}(\pi)$. For example, let $\pi = 351624$, which is alternating. Its right-to-left maxima are $4, 6$, the right-to-left minima are $4, 2, 1$, the left-to-right maxima are $3, 5, 6$, and the left-to-right minima are $3, 1$. Hence $\mathrm{rmax}(\pi) = 2$, $\mathrm{rmin}(\pi) = 3$, $\mathrm{lmax}(\pi) = 3$, and $\mathrm{lmin}(\pi) = 2$.

Theorem 3.2. *All patterns in Class 2 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the first pattern in each of the seven blocks of Class 2. It is enough to prove that these seven representatives are equidistributed on $\mathcal{A}_n$:



p_1 p_2 p_3 p_4 p_5 p_6 p_7

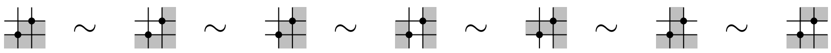
Fix $\pi \in \mathcal{A}_n$. Since both 1 and π_n are right-to-left minima, there are $\mathrm{rmin}(\pi) - 1$ right-to-left minima different from 1, as well as $\mathrm{rmin}(\pi) - 1$ nonfinal right-to-left minima. Examining the shaded boxes from left to right yields the following bijections between occurrences:

- p_1 pairs 1 with each right-to-left minimum different from 1;
- p_2 pairs consecutive right-to-left minima in their left-to-right order;
- p_3 pairs each right-to-left minimum $b \neq 1$ with the leftmost element smaller than b;
- p_4 pairs each right-to-left minimum $b \neq 1$ with the element $b - 1$;
- p_5, p_6, and p_7 pair each nonfinal right-to-left minimum with, respectively, the final element, the next element, and the largest element to its right.

In each case, the corresponding partner is unique, and the shading conditions characterize precisely the indicated pair. Conversely, every such pair determines an occurrence. Hence $p_i(\pi) = \mathrm{rmin}(\pi) - 1$ for $1 \leq i \leq 7$. Thus the seven statistics agree pointwise on $\mathcal{A}_n$ and, in particular, are equidistributed. $\square$

Theorem 3.3. *All patterns in Class 3 are equidistributed on $\mathcal{A}_n$.*

Proof. Choosing the first pattern from each of the seven blocks in Class 3, it suffices to compare the following seven representatives:



p_1 p_2 p_3 p_4 p_5 p_6 p_7

Fix $\pi \in \mathcal{A}_n$. Reading the shaded boxes gives the unique possible occurrence of each representative:

- p_1, p_2, p_6, and p_7 pair 1 with, respectively, the smallest element to its right, the final element, the next element, and the largest element to its right;
- p_3 and p_4 pair π_n with, respectively, the rightmost and leftmost earlier element smaller than π_n;
- p_5 pairs π_n with the earlier element $\pi_n - 1$.

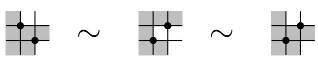
Each indicated partner exists precisely when $\pi_n \neq 1$, and the shaded boxes characterize exactly that pair. Hence, for $1 \leq i \leq 7$,

$$p_i(\pi) = \begin{cases} 1, & \pi_n \neq 1, \\ 0, & \pi_n = 1. \end{cases}$$

Thus the seven representatives are pointwise equal on $\mathcal{A}_n$, and therefore all patterns in Class 3 are equidistributed. $\square$

Theorem 3.4. *All patterns in Class 4 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representatives from the first, second, and third blocks of Class 4, respectively. It is enough to prove their equivalence:



p_1 p_2 p_3

Each pattern has at most one occurrence. The shaded boxes give the following conditions:

$$\begin{aligned}
p_1:&\quad \pi_1 > 1 \text{ and } \pi_1 - 1 \text{ precedes every value smaller than } \pi_1 - 1,\\
p_2:&\quad \pi_1 < n \text{ and } n \text{ precedes every value smaller than } \pi_1,\\
p_3:&\quad \pi_1 < n \text{ and } \pi_1 + 1 \text{ precedes every value smaller than } \pi_1.
\end{aligned}$$

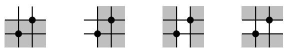
For each fixed $\pi_1 = a < n$, the one-occurrence sets for p_2 and p_3 are $\mathcal{E}^{\text{pre}}_{a,n}$ and $\mathcal{E}^{\text{pre}}_{a,a+1}$, respectively. Applying Lemma 2.2 therefore gives $p_2 \sim p_3$.

It remains to compare p_1 and p_3. Suppose that π has a p_3-occurrence. If π_1 and $\pi_1 + 1$ are not adjacent, interchange these two values. The resulting alternating permutation begins with $\pi_1 + 1$, and the value π_1 precedes every smaller value, so it has a p_1-occurrence. If π_1 and π_1+1 are adjacent, then they are the first two elements; complementing π changes this initial ascent into an initial consecutive descent and again gives a p_1-occurrence. The inverse uses the same value interchange in the nonadjacent case and complement in the adjacent case. This is a bijection between the one-occurrence sets of p_3 and p_1, so $p_1 \sim p_3$.

Hence $p_1 \sim p_2 \sim p_3$, proving the class. □

Theorem 3.5. *All patterns in Class 5 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representative from each of the four blocks of Class 5:



p_1 p_2 p_3 p_4

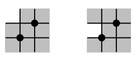
It is enough to prove $p_1 \sim p_2 \sim p_3 \sim p_4$. Each representative has at most one occurrence on $\mathcal{A}_n$. Reading the shaded boxes gives

$$\begin{aligned}
p_1:&\quad \mathrm{pos}_\pi(1) < \mathrm{pos}_\pi(2),\\
p_2:&\quad \pi_{n-1} < \pi_n,\\
p_3:&\quad \pi_1 < \pi_n,\\
p_4:&\quad \mathrm{pos}_\pi(1) < \mathrm{pos}_\pi(n).
\end{aligned}$$

For $n \geq 2$, each displayed event holds for exactly half of $\mathcal{A}_n$. Indeed, reverse interchanges the two possible orders of 1 and 2, and also interchanges the inequalities $\pi_1 < \pi_n$ and $\pi_1 > \pi_n$. Complement interchanges a final ascent with a final descent, and interchanges the two possible orders of 1 and n. Both maps are bijections on $\mathcal{A}_n$. Hence the four one-occurrence sets have cardinality $|\mathcal{A}_n|/2$, so $p_1 \sim p_2 \sim p_3 \sim p_4$. □

Theorem 3.6. *All patterns in Class 6 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representative from each block of Class 6:



p_1 p_2

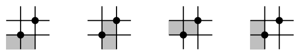
It is enough to prove $p_1 \sim p_2$. The shaded boxes give

$$p_1(\pi) = 1 \iff \pi_{n-1}\pi_n = 12, \qquad p_2(\pi) = 1 \iff \pi_{n-1}\pi_n = 1n.$$

The last assertion of Lemma 2.2 shows that equally many alternating permutations end in 12 and in $1n$. Since both statistics take only the values 0 and 1, we obtain $p_1 \sim p_2$. $\square$

Theorem 3.7. *All patterns in Class 7 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representative from each block of Class 7:



p_1 p_2 p_3 p_4

It is enough to prove $p_1 \sim p_2 \sim p_3 \sim p_4$.

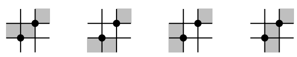
Fix $\pi \in \mathcal{A}_n$, and let j be a position such that π_j is not a left-to-right minimum. Then there is at least one element to the left of π_j that is smaller than π_j. Among these elements, one may choose, respectively, the smallest element, the rightmost element, the largest element, or the leftmost element. Each of these four choices is unique and determines a unique partner for π_j. The shading conditions in the four patterns characterize precisely these four possible choices. Hence

$$p_1(\pi) = p_2(\pi) = p_3(\pi) = p_4(\pi) = n - \mathrm{lmin}(\pi),$$

which proves the result. $\square$

Theorem 3.8. *All patterns in Class 8 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representative from each block of Class 8:



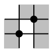
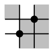
p_1 p_2 p_3 p_4

It is enough to prove $p_1 \sim p_2 \sim p_3 \sim p_4$.

Fix $\pi \in \mathcal{A}_n$. For each right-to-left maximum b having a smaller element to its left, the four patterns pair b with, respectively, the largest, smallest, leftmost, and rightmost such element. Each partner is unique, and the shaded boxes characterize exactly these four choices. Consequently, $p_1(\pi) = p_2(\pi) = p_3(\pi) = p_4(\pi)$, so the four representatives are pointwise equal. $\square$

Theorem 3.9. *All patterns in Class 9 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representative from each block of Class 9:

p_1 $\quad$ p_2

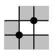
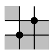
It is enough to prove $p_1 \sim p_2$. In both patterns, the fully shaded rightmost column forces the second selected element to be $b = \pi_n$. For p_1, the first selected element must be the first element smaller than b; for p_2, it must be the last element smaller than b. Thus each pattern has at most one occurrence, and the shaded boxes give

$$\begin{aligned} p_1: &\quad \pi = A\,B\,b, \quad A > b > B, \quad B \neq \emptyset, \\ p_2: &\quad \pi = B\,A\,b, \quad B < b < A, \quad B \neq \emptyset. \end{aligned}$$

If $b = n$, both patterns occur exactly once: the occurrences are $\pi_1 n$ for p_1 and $\pi_{n-1} n$ for p_2. Thus the identity map matches these one-occurrence sets, although the selected pairs are different. If $b < n$, both A and B are nonempty, and complement maps the first decomposition to the second. Hence, using the identity map when $b = n$ and complement when $b < n$ gives a bijection between the one-occurrence sets. Since both statistics take only the values 0 and 1, we obtain $p_1 \sim p_2$. □

Theorem 3.10. *All patterns in Class 10 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representative from each block of Class 10:

p_1 $\quad$ p_2

It is enough to prove $p_1 \sim p_2$. In both patterns, the fully shaded bottom row forces the first selected element to be 1. Writing $\pi = A\,1\,B$, the shaded boxes give

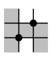
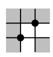
$$\begin{aligned} p_1: &\quad B \neq \emptyset, \quad A > B, \quad \text{with occurrence given by 1 and } \max B, \\ p_2: &\quad B \neq \emptyset, \quad A < B, \quad \text{with occurrence given by 1 and } \min B. \end{aligned}$$

Thus both statistics take only the values 0 and 1.

For $\pi = A\,1\,B$ in the first one-occurrence set, relabel the elements of A and B, preserving the standardization of each block, so that the resulting blocks satisfy $A' < B'$. The comparisons within each block are unchanged, and the comparisons adjacent to 1 are also unchanged. Hence $A'\,1\,B'$ is alternating and has one p_2-occurrence. This map is reversible, so the two one-occurrence sets have the same size. Hence $p_1 \sim p_2$. □

Theorem 3.11. *All patterns in Class 11 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representatives from the two blocks of Class 11. It is enough to prove

 $\sim$ 

p_1 $\qquad$ p_2

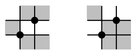
Each pattern has at most one occurrence. The shaded boxes give

$$\begin{aligned}
p_1 :\quad & \pi_1 < n \text{ and } \pi_1 + 1 \text{ follows every value smaller than } \pi_1,\\
p_2 :\quad & \pi_1 < n \text{ and } n \text{ follows every value smaller than } \pi_1.
\end{aligned}$$

This is the same comparison as that between p_2 and p_3 in the proof of Theorem 3.4, with "precedes" replaced by "follows." For each fixed $\pi_1 = a < n$, the one-occurrence sets are $\mathcal{E}^{\text{post}}_{a,a+1}$ and $\mathcal{E}^{\text{post}}_{a,n}$, respectively. Lemma 2.2 gives $|\mathcal{E}^{\text{post}}_{a,a+1}| = |\mathcal{E}^{\text{post}}_{a,n}|$ for every a. Hence the two one-occurrence sets, after taking the disjoint union over $1 \le a < n$, have the same size. Since both statistics take only the values 0 and 1, we conclude that $p_1 \sim p_2$. □

Theorem 3.12. *All patterns in Class 12 are equidistributed on $\mathcal{A}_n$.*

Proof. Choose the following representative from each block of Class 12:



p_1 $\quad$ p_2

It is enough to prove $p_1 \sim p_2$. For $1 \le i < n$, the shaded boxes give the following descriptions of an occurrence starting at position i:

$$\begin{aligned}
p_1 :\quad & \{\pi_i, \dots, \pi_n\} \text{ is an interval of consecutive values,}\\
& \pi_i = \min\{\pi_i, \dots, \pi_n\};\\
p_2 :\quad & \{\pi_1, \dots, \pi_i\} = \{1, \dots, i\}.
\end{aligned}$$

Each occurrence is uniquely determined by the position of its first point. For $j \in \{1, 2\}$ and $\pi \in \mathcal{A}_n$, let

$$O_j(\pi) := \{i \in \{1, \dots, n-1\} : \text{a } p_j\text{-occurrence in } \pi \text{ begins at position } i\}.$$

For fixed $k \ge 0$, define

$$\begin{aligned}
\mathcal{X}_k &:= \{(\pi, I) : \pi \in \mathcal{A}_n,\ I \subseteq O_1(\pi),\ |I| = k\},\\
\mathcal{Y}_k &:= \{(\sigma, I) : \sigma \in \mathcal{A}_n,\ I \subseteq O_2(\sigma),\ |I| = k\}.
\end{aligned}$$

Thus $\mathcal{X}_k$ is the full set of alternating permutations together with a choice of k distinct p_1-occurrences, and $\mathcal{Y}_k$ is the corresponding set for p_2. We construct mutually inverse maps

$$F : \mathcal{X}_k \longrightarrow \mathcal{Y}_k, \qquad G : \mathcal{Y}_k \longrightarrow \mathcal{X}_k.$$

For $k = 0$, both maps are the identity. Suppose $k \ge 1$, take $(\pi, I) \in \mathcal{X}_k$, write $I = \{i_1 < \cdots < i_k\}$, and cut π immediately after these positions:

$$\pi = C_0 \mid C_1 \mid \cdots \mid C_k.$$

Replace the elements of each block, without changing their relative order, by the consecutive value sets

$$\{1, \dots, i_1\}, \quad \{i_1 + 1, \dots, i_2\}, \quad \dots \quad , \{i_k + 1, \dots, n\}.$$

Denote the resulting permutation by σ. The prefix ending at i_j has value set $\{1, \ldots, i_j\}$ for every j, so σ_{i_j} is the first point of a p_2-occurrence. Thus the image is $F(\pi, I) = (\sigma, I)$. Moreover, the relative order inside each block is unchanged. At every cut the element on the left is smaller than the element on the right, because a p_1-occurrence begins with the minimum of its suffix. Hence $\sigma \in \mathcal{A}_n$, and F is well defined.

To define G, start with $(\sigma, I) \in \mathcal{Y}_k$ and cut σ after the positions in I. Since every σ_{i_j} is the first point of a p_2-occurrence, the blocks have exactly the consecutive value sets displayed above. Thus σ determines the relative order within every block, which is the information retained by the forward map.

We now restore the actual labels. Set $R_0 = \{1, \ldots, n\}$. For $0 \le j < k$, suppose that the consecutive interval R_j is the set of values available for $C_j \cdots C_k$. Put $N_j = |R_j|$ and $d_j = |C_j|$, and let s_j be the rank of the last element of C_j in the known relative order of that block. Let

$$\rho_j : \{1, \ldots, N_j\} \longrightarrow R_j$$

be the unique increasing bijection. In a preimage, the last element of C_j must be the minimum of a consecutive suffix of length $N_j - d_j + 1$. Exactly $s_j - 1$ elements of C_j precede it in value, so its standardized value is forced to be s_j. Consequently, the values for the later blocks are forced to be the actual interval

$$R_{j+1} := \rho_j\big(\{s_j + 1, \ldots, s_j + N_j - d_j\}\big).$$

Fill C_j, in its known relative order, with the remaining actual values $R_j \setminus R_{j+1}$. After completing these steps for $j = 0, \ldots, k-1$, fill C_k with the values of R_k in its known relative order; call the resulting permutation τ.

At the cut after C_j, the last element of C_j is $\rho_j(s_j)$, and the suffix beginning there has the consecutive value set

$$\rho_j\big(\{s_j, s_j + 1, \ldots, s_j + N_j - d_j\}\big).$$

Thus every position in I begins a p_1-occurrence. The relative order inside each block is the same as in σ, and every cut is an ascent because all values of R_{j+1} exceed $\rho_j(s_j)$. Hence $\tau \in \mathcal{A}_n$, and we may define $G(\sigma, I) = (\tau, I)$.

The preceding reconstruction is forced at every step. Since F preserves all block relative orders, applying G to $F(\pi, I)$ restores the original value set of every block and hence $G(F(\pi, I)) = (\pi, I)$. Conversely, G preserves the block relative orders of σ, and F replaces their value sets by the consecutive sets displayed above, which are exactly the value sets of the blocks of σ. Hence $F(G(\sigma, I)) = (\sigma, I)$. Therefore F and G are mutual inverses. In particular, if $F(x) = F(x')$ for $x, x' \in \mathcal{X}_k$, then applying G gives $x = x'$, so F is injective. For every $y \in \mathcal{Y}_k$, the element $G(y)$ belongs to $\mathcal{X}_k$ and satisfies $F(G(y)) = y$, so F is surjective. Hence $F : \mathcal{X}_k \to \mathcal{Y}_k$ is a bijection.

For example, take $I = \{2\}$ and $\pi = 51423$. Then

$$51423 = 51 \mid 423 \ \longmapsto\ 21 \mid 534 = 21534.$$

The two blocks keep their relative orders and receive the value sets $\{1, 2\}$ and $\{3, 4, 5\}$. Conversely, in $21 \mid 534$ the last element of 21 is the smallest element of its block. It must therefore be 1, and the suffix beginning at position 2 must use $\{1, 2, 3, 4\}$. The only remaining value 5 goes to the first element, restoring 51; assigning $\{2, 3, 4\}$ to the second block in its relative order restores 423. Hence the inverse recovers $51 \mid 423 = 51423$.

For a fixed π and $i \in \{1, 2\}$, there are $\binom{p_i(\pi)}{k}$ ways to choose k occurrences of p_i in π. Hence the two sides below count the source and target pairs, respectively, of the bijection above. Thus, for every $k \geq 0$,

$$\sum_{\pi \in \mathcal{A}_n} \binom{p_1(\pi)}{k} = \sum_{\pi \in \mathcal{A}_n} \binom{p_2(\pi)}{k}. \tag{1}$$

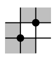
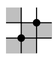
Multiplying both sides of (1) by $(q-1)^k$, then summing over all $k \geq 0$ and using the binomial theorem, we obtain

$$\sum_{\pi \in \mathcal{A}_n} q^{p_1(\pi)} = \sum_{\pi \in \mathcal{A}_n} q^{p_2(\pi)},$$

so p_1 and p_2 have the same distribution on $\mathcal{A}_n$. $\square$

Theorem 3.13. *All patterns in Class 13 are equidistributed on $\mathcal{A}_n$.*

Proof. It is enough to compare the following representatives of the two blocks:

$p_1 =$  and $p_2 =$ .

We first describe the map for one marked occurrence. Suppose that its first point is a in position i, and put $r = i - 1$. For a p_1-occurrence, the r elements preceding a are precisely the values larger than its second point. Hence that point is $b = n - r$, and we may write $\pi = L\, S$, where L consists of the r elements in $\{b+1, \dots, n\}$, and S begins with a. Within S, the occurrence condition says exactly that b precedes every value smaller than a. Moreover, the first comparison in S is an ascent: the element following a is either the selected value b or an element lying between the two selected positions, and in either case it is larger than a.

Apply successively $\theta_{b-1}, \theta_{b-2}, \dots, \theta_{a+1}$ to S. By Lemma 2.2, this is a bijection that changes the last condition to "$a + 1$ precedes every value smaller than a." Call the resulting suffix S'. Since every toggle preserves adjacent comparisons, the first comparison in S' is also an ascent. Now relabel L, preserving its relative order, with the value set $\{a+1, a+2, \dots, a+r\}$, and in S' replace every value $x > a$ by $x + r$, leaving the values $x \leq a$ fixed. Let the resulting permutation be σ.

The first marked point is still a, whereas the marked value $a + 1$ in S' has become $c = a + r + 1$. All values strictly between a and c now precede a; every element between a and c in position is larger than c; and every value smaller than a follows c. These are exactly the shading conditions for a p_2-occurrence. The toggle step preserves alternation. The relabeling preserves all comparisons inside L and S', and it preserves the comparison at their boundary because every element of the new L is still larger than a. Thus $\sigma \in \mathcal{A}_n$.

For completeness, the inverse is explicit. From a marked p_2-occurrence beginning with a in position i, put $r = i - 1$ and $c = a + r + 1$. Compress its suffix by replacing $x \geq c$ with $x - r$, and relabel its prefix, preserving relative order, with $\{n-r+1, \ldots, n\}$. Finally apply $\theta_{a+1}, \theta_{a+2}, \ldots, \theta_{n-r-1}$ to the compressed suffix. This reverses the preceding construction and recovers the marked p_1-occurrence.

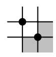
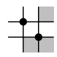
Distinct occurrences of either pattern have disjoint position intervals. Moreover, if two p_1-occurrences are ordered from left to right, both selected values of the second are smaller than the first selected value of the first. Thus the toggles for the first occurrence do not touch the value layer of the second, while the later block relabeling is order-preserving on every earlier value layer. Therefore, for a chosen set of marked occurrences, apply the forward construction in increasing order of their first positions; all unprocessed p_1-occurrences and all previously obtained p_2-occurrences remain valid. Apply the inverse construction in decreasing order. Hence, for every k, this gives a bijection between pairs (π, M), where $\pi \in \mathcal{A}_n$ and M is a chosen set of k distinct p_1-occurrences of π, and the corresponding pairs for p_2. Consequently,

$$\sum_{\pi \in \mathcal{A}_n} \binom{p_1(\pi)}{k} = \sum_{\pi \in \mathcal{A}_n} \binom{p_2(\pi)}{k} \qquad (k \geq 0),$$

which implies that p_1 and p_2 are equidistributed. $\square$

Theorem 3.14. *All patterns in Class 14 are equidistributed on $\mathcal{A}_n$.*

Proof. By reverse, complement, and Lemma 2.1, it is enough to compare the following two 21-patterns:

$p_1 =$  and $p_2 =$ .

We use the involution Ψ of Han and Zeng [9, Theorem 1.9]. Recall only the part of their construction needed here. Let $i_1 < i_2 < \cdots < i_\ell = n$ be the positions of the right-to-left minima of a permutation, and set $i_0 = 0$. Write

$$B_k = \pi_{i_{k-1}+1}\pi_{i_{k-1}+2} \cdots \pi_{i_k-1}, \qquad a_k = \pi_{i_k}.$$

To complement a subword with alphabet $c_1 < \cdots < c_r$, replace each c_j by c_{r+1-j}. For each a_k, Han and Zeng first complement the subword of letters larger than a_k lying to the right of i_{k-1}, and then the corresponding subword lying to the right of i_k. Applying these two operations successively for $k = 1, \ldots, \ell$ defines Ψ and preserves the positions of the right-to-left minima. The first component of Han and Zeng's joint identity in [9, Theorem 1.9] is precisely

$$p_2(\pi) = p_1(\Psi(\pi)) \qquad (\pi \in S_n).$$

Their construction realizes this identity occurrence by occurrence: if (j, i_k) is a p_2-occurrence of π, then the same two positions form a p_1-occurrence of $\Psi(\pi)$, and every p_1-occurrence of $\Psi(\pi)$ arises uniquely in this way.

This involution does not preserve alternation. For example, $\pi = 25143 \in \mathcal{A}_5$ has right-to-left minima in positions $i_1 = 3$ and $i_2 = 5$. At i_1, the two complementations give

$$25143 \longmapsto 52134 \longmapsto 52143,$$

and the operation at i_2 is trivial. Hence $\Psi(25143) = 52143 \notin \mathcal{A}_5$. Thus the original involution of Han and Zeng cannot be restricted to $\mathcal{A}_n$.

For our two patterns this action can be read directly from the shaded boxes. An occurrence ending at the right-to-left minimum a_k has its first point in B_k. For $x \in B_k$, the pair (x, a_k) is a p_1-occurrence precisely when every element to the right of a_k is larger than x, while it is a p_2-occurrence precisely when every element to the right of a_k is smaller than x. Thus the occurrence count in the kth block depends only on the value set of B_k and the value set to the right of a_k, not on the internal order of B_k. This is exactly the local comparison interchanged by Han and Zeng's complementation.

The map Ψ gives the required change from p_2-occurrences to p_1-occurrences, but it may reverse comparisons within a block and hence destroy alternation. The idea is to keep the new value set that Ψ assigns to each block, while restoring the relative order of the corresponding block of π. We now define this modification precisely.

Given $\pi \in \mathcal{A}_n$, first obtain $\tau = \Psi(\pi)$ by performing the two complementations above for $k = 1, \ldots, \ell$. For each k, write $B_k = b_1 \cdots b_r$, and let C_k be the word occupying positions $i_{k-1} + 1, \ldots, i_k - 1$ in τ. If b_t is the sth smallest element of B_k, put the sth smallest element of C_k in the tth position of that block. Do this in every block and leave the elements of τ in positions $i_1, \ldots, i_\ell$ unchanged; the resulting permutation is $\widehat{\Psi}(\pi)$. For example, when $\pi = 25143$, the first block 25 becomes 52 in $\Psi(\pi) = 52143$; reordering 52 to have the same relative order as 25 gives $\widehat{\Psi}(\pi) = 25143$.

Thus every block of $\widehat{\Psi}(\pi)$ has the value set assigned by Ψ and the relative order of the corresponding block of π. Comparisons inside each block are therefore the same as in π. Moreover, since $i_1, \ldots, i_\ell$ are also the right-to-left-minimum positions of τ, every element of C_k is larger than the adjacent right-to-left minima, whenever those boundary elements exist. Hence the boundary comparisons are also unchanged, and $\widehat{\Psi}(\pi) \in \mathcal{A}_n$.

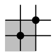
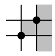
It remains to check that this reordering preserves the p_1-occurrences. Let (a, b) be a p_1-occurrence of τ, with b in position i_k. Then a lies in C_k, and every element to the right of b is larger than a. The reordering may move a to another position inside C_k, but it does not change b or the value set to the right of b; moreover, every element of C_k is larger than b. Hence the same two values a and b still form a p_1-occurrence of $\widehat{\Psi}(\pi)$. The reverse reordering gives the converse, so

$$p_2(\pi) = p_1(\Psi(\pi)) = p_1(\widehat{\Psi}(\pi)).$$

Finally, $\widehat{\Psi}$ is bijective. Indeed, each block of $\widehat{\Psi}(\pi)$ records exactly two pieces of information: its value set is the value set of the corresponding block of $\Psi(\pi)$, and its relative order is that of the original block B_k. Since Ψ is a bijection and its complementations act on block value sets independently of their internal order, Ψ^{-1} uniquely recovers the value set of every B_k. Arranging each recovered set in the relative order shown by the corresponding block of $\widehat{\Psi}(\pi)$ then recovers π uniquely. Therefore $\widehat{\Psi}$ is a bijection on $\mathcal{A}_n$, and consequently $p_1 \sim p_2$ on $\mathcal{A}_n$. □

Theorem 3.15. *All patterns in Class 15 are equidistributed on $\mathcal{A}_n$.*

Proof. By the preceding observation, it is enough to compare

$p_1 =$  and $p_2 =$ .

The statistic $p_1(\pi)$ counts the elements that are second smallest in their prefixes. A p_2-occurrence must end at π_n, so $p_2(\pi)$ counts the right-to-left maxima in the subword of elements below π_n.

We prove a descent-set refinement. Let $\alpha = (a_1, \dots, a_m)$ be a composition of n, put

$$D(\alpha) = \{a_1, a_1 + a_2, \dots, a_1 + \cdots + a_{m-1}\},$$

and set

$$\mathcal{S}_\alpha := \{\pi \in S_n : \mathrm{Des}(\pi) \subseteq D(\alpha)\}.$$

Let $\mathcal{W}_\alpha$ be the set of all words consisting of a_1 copies of 1, a_2 copies of 2, …, and a_m copies of m. Every $\pi \in \mathcal{S}_\alpha$ is uniquely a concatenation $B_1 \cdots B_m$ of increasing blocks with $|B_i| = a_i$. Define

$$\varphi_\alpha : \mathcal{S}_\alpha \longrightarrow \mathcal{W}_\alpha, \qquad \varphi_\alpha(\pi) = w_1 \cdots w_n, \qquad w_j = i \iff j \in B_i.$$

Conversely, a word $w \in \mathcal{W}_\alpha$ recovers B_i as the values j with $w_j = i$, written in increasing order. The concatenation $B_1 \cdots B_m$ then belongs to $\mathcal{S}_\alpha$ and maps to w. Hence φ_α is a bijection.

For $w \in \mathcal{W}_\alpha$, let $r_\alpha(w)$ be the number of indices i for which the subword $w|_{\{1,\dots,i\}}$ has a second letter and that letter is i. Indeed, the second letter of this subword records the block containing the second-smallest value in $B_1 \cup \cdots \cup B_i$. It is i precisely when this value is the unique element of B_i that is second smallest in its prefix. Therefore

$$p_1(\pi) = r_\alpha\big(\varphi_\alpha(\pi)\big) \qquad (\pi \in \mathcal{S}_\alpha).$$

Since φ_α is a bijection, it follows that

$$P_\alpha(q) := \sum_{\pi \in \mathcal{S}_\alpha} q^{p_1(\pi)} = \sum_{w \in \mathcal{W}_\alpha} q^{r_\alpha(w)}. \tag{2}$$

Suppose $m \geq 2$ and set $\alpha^- = (a_1, \dots, a_{m-1})$. Every $w \in \mathcal{W}_\alpha$ is obtained uniquely from a word $w' \in \mathcal{W}_{\alpha^-}$ by inserting a_m copies of m. For each $i < m$, the restriction $w|_{\{1,\dots,i\}}$ automatically deletes all inserted letters m and hence equals $w'|_{\{1,\dots,i\}}$. Thus i contributes equally to $r_\alpha(w)$ and $r_{\alpha^-}(w')$. For $i = m$, the restricted subword is w itself, and this index contributes exactly when $w_2 = m$. Hence

$$r_\alpha(w) = \begin{cases} r_{\alpha^-}(w') + 1, & \text{if } w_2 = m, \\ r_{\alpha^-}(w'), & \text{if } w_2 \neq m. \end{cases}$$

There are $\binom{n-1}{a_m-1}$ insertions with m in the second position and $\binom{n-1}{a_m}$ with it elsewhere. Summing the contributions of all insertions over all w' in (2) gives

$$P_\alpha(q) = \left[\binom{n-1}{a_m} + q\binom{n-1}{a_m-1}\right] P_{\alpha^-}(q). \tag{3}$$

For p_2, set $\beta = \alpha^{\mathrm{rev}} = (a_m, \dots, a_1)$. Define $\mathcal{S}_\beta$, $\mathcal{W}_\beta$, and the bijection $\varphi_\beta : \mathcal{S}_\beta \to \mathcal{W}_\beta$ analogously. The index of the last letter m in such a word is the value π_n. For $x < \pi_n$, any larger value $y < \pi_n$ lies to the right of x in the permutation exactly when $w_y \geq w_x$, since each block is increasing. Thus x is a right-to-left maximum among the elements below π_n precisely when $w_x > w_y$ for every $x < y < \pi_n$. Hence, if $s_\beta(w)$ denotes the number of strict right-to-left maxima in the prefix preceding the last m, then

$$Q_\beta(q) := \sum_{\pi \in \mathcal{S}_\beta} q^{p_2(\pi)} = \sum_{w \in \mathcal{W}_\beta} q^{s_\beta(w)}. \tag{4}$$

The smallest letter in a word of content β occurs a_m times. Delete these copies of 1 and decrease every remaining letter by one. The resulting word has content

$$\beta^- = (a_{m-1}, \dots, a_1) = (\alpha^-)^{\mathrm{rev}}.$$

Conversely, increase every letter of a word in $\mathcal{W}_{\beta^-}$ by one and reinsert the a_m copies of 1. The old strict right-to-left maxima are unchanged. A new one is created exactly when the gap immediately before the last m is nonempty; in that case it is the rightmost 1 in this gap. Since the shortened word has $n - a_m + 1$ gaps, reserving one 1 in this distinguished gap and distributing the others gives $\binom{n-1}{a_m-1}$ nonempty cases. Keeping it empty and using the remaining gaps gives $\binom{n-1}{a_m}$ cases. Summing as in (4) gives

$$Q_{\alpha^{\mathrm{rev}}}(q) = \left[\binom{n-1}{a_m} + q\binom{n-1}{a_m - 1}\right] Q_{(\alpha^-)^{\mathrm{rev}}}(q). \tag{5}$$

The recurrences (3) and (5) have the same factor, say $F_\alpha(q)$. When $m = 1$, the only word is 1^{a_1}, so both initial polynomials equal q if $a_1 \geq 2$ and 1 if $a_1 = 1$. Assuming the result for α^-, equations (3) and (5) give

$$P_\alpha(q) = F_\alpha(q) P_{\alpha^-}(q) = F_\alpha(q) Q_{(\alpha^-)^{\mathrm{rev}}}(q) = Q_{\alpha^{\mathrm{rev}}}(q).$$

Thus induction on m proves $P_\alpha(q) = Q_{\alpha^{\mathrm{rev}}}(q)$.

Every $D \subseteq \{1, \dots, n-1\}$ is $D(\alpha)$ for a unique composition α, and

$$D(\alpha^{\mathrm{rev}}) = D(\alpha)^*, \qquad D^* := \{n - i : i \in D\}.$$

We have therefore proved

$$\sum_{\mathrm{Des}(\pi) \subseteq D} q^{p_1(\pi)} = \sum_{\mathrm{Des}(\pi) \subseteq D^*} q^{p_2(\pi)}. \tag{6}$$

Inclusion–exclusion applied to (6) yields

$$\sum_{\mathrm{Des}(\pi) = D} q^{p_1(\pi)} = \sum_{\mathrm{Des}(\pi) = D^*} q^{p_2(\pi)}. \tag{7}$$

The cases $n \le 1$ are immediate. Assume $n \ge 2$ and set

$$D_{\mathrm{UD}} := \{2, 4, \ldots\} \cap \{1, \ldots, n-1\}, \qquad D_{\mathrm{DU}} := \{1, 3, \ldots\} \cap \{1, \ldots, n-1\}.$$

Then

$$UD_n = \{\pi \in S_n : \mathrm{Des}(\pi) = D_{\mathrm{UD}}\}, \qquad DU_n = \{\pi \in S_n : \mathrm{Des}(\pi) = D_{\mathrm{DU}}\}.$$

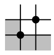
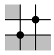
If n is even, then $D^*_{\mathrm{UD}} = D_{\mathrm{UD}}$ and $D^*_{\mathrm{DU}} = D_{\mathrm{DU}}$; if n is odd, then $D^*_{\mathrm{UD}} = D_{\mathrm{DU}}$ and $D^*_{\mathrm{DU}} = D_{\mathrm{UD}}$. Applying (7) to D_{UD} and D_{DU} and adding the two identities therefore gives

$$\sum_{\pi \in \mathcal{A}_n} q^{p_1(\pi)} = \sum_{\pi \in \mathcal{A}_n} q^{p_2(\pi)}.$$

Thus p_1 and p_2 are equidistributed on $\mathcal{A}_n$. □

Theorem 3.16. *All patterns in Class 16 are equidistributed on $\mathcal{A}_n$.*

Proof. By reverse and complement, it is enough to compare

$p_1 =$  and $p_2 =$ .

Recall that $E_s = |UD_s| = |DU_s|$. Write $\pi = L\,1\,R$ and let $m = |L|$. The fully shaded bottom row forces the first point of every occurrence to be the value 1. If $L = \varnothing$, every other value can be the second point of either pattern, so $p_1(\pi) = p_2(\pi) = n-1$.

Now suppose $L \neq \varnothing$. If $a = \min L$, then the possible second points of a p_1-occurrence are precisely $2, 3, \ldots, a-1$; hence $p_1(\pi) = a-2$. If $c = \max L$, then the possible second points of a p_2-occurrence are precisely $c+1, c+2, \ldots, n$; hence $p_2(\pi) = n-c$.

Since 1 is a valley, the alternating type of L is determined by m, while R is down-up. Thus L has E_m possible orders and R has E_{n-m-1} possible orders. These choices are independent, giving $E_m E_{n-m-1}$ orders in total. For fixed $a = \min L$, the other $m-1$ values of L can be chosen from $\{a+1, \ldots, n\}$; therefore

$$\sum_{\pi \in \mathcal{A}_n} q^{p_1(\pi)} = E_{n-1} q^{n-1} + \sum_{m=1}^{n-1} \sum_{a=2}^{n-m+1} \binom{n-a}{m-1} E_m E_{n-m-1} q^{a-2}. \tag{8}$$

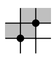
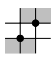
Similarly, fixing $c = \max L$ gives

$$\sum_{\pi \in \mathcal{A}_n} q^{p_2(\pi)} = E_{n-1} q^{n-1} + \sum_{m=1}^{n-1} \sum_{c=m+1}^{n} \binom{c-2}{m-1} E_m E_{n-m-1} q^{n-c}.$$

The substitution $a = n+2-c$ makes the two double sums identical. Thus p_1 and p_2 are equidistributed on $\mathcal{A}_n$. □

Theorem 3.17. *All patterns in Class 17 are equidistributed on $\mathcal{A}_n$.*

Proof. By reverse, complement, and Lemma 2.1, it is enough to compare

$p_1 =$  and $p_2 =$ .

Write $\pi = A_1 r_1 A_2 r_2 \cdots A_\ell r_\ell$, where $r_1 > \cdots > r_\ell$ are the right-to-left maxima. Let N_i be the set of elements in A_i, and let M_i be the set of elements smaller than r_i that lie to the left of r_{i-1}; set $M_1 = \varnothing$. Thus $U_i = M_i \cup N_i$ is the set of elements smaller than r_i and to its left. The common top shading forces the second point of an occurrence to be r_i and the first point to lie in A_i. Let $p_{1,i}(\pi)$ and $p_{2,i}(\pi)$ denote the numbers of p_1- and p_2-occurrences, respectively, whose second point is r_i. The remaining shading gives

$$p_{1,i}(\pi) = \mathbf{1}_{\{\max U_i \in N_i\}}, \qquad p_{2,i}(\pi) = \mathbf{1}_{\{\min U_i \in N_i\}}, \tag{9}$$

where $\mathbf{1}_E$ is 1 if E holds and 0 otherwise; both indicators are zero when $U_i = \varnothing$. Consequently $p_1(\pi) = \sum_i p_{1,i}(\pi)$ and $p_2(\pi) = \sum_i p_{2,i}(\pi)$.

We now define an involution T_i. Write $U_i = \{u_1 < \cdots < u_m\}$ and form a word $w = w_1 \cdots w_m$ by setting $w_k = M$ if $u_k \in M_i$ and $w_k = N$ if $u_k \in N_i$. Reverse w. For each k, put u_k in M_i' or N_i' according as the kth letter of the reversed word is M or N. Put the values of M_i' into the positions formerly occupied by M_i, preserving their relative order, and do the same with N_i' in the positions of A_i; leave all other elements fixed. For example, for $\pi = 3172546$ and $i = 2$,

$$M_2 = \{1, 3\}, \qquad N_2 = \{2, 4, 5\}, \qquad MNMNN \longmapsto NNMNM,$$

and hence $T_2(3172546) = 5371426$.

The map T_i preserves every adjacent comparison. Inside each of the two changed position sets this follows from the order-preserving refilling. Every unchanged neighbor of a changed element to the left of r_{i-1} is larger than r_i, while every changed element is smaller than r_i; the boundary elements r_{i-1} and r_i of A_i, when present, are also larger than all changed elements. Thus T_i preserves alternation. It also fixes all right-to-left maxima and is an involution, since reversing the same M/N word twice restores the original value sets and relative orders.

By (9), reversing the word interchanges its first and last letters, so

$$p_{1,i}(T_i\pi) = p_{2,i}(\pi), \qquad p_{2,i}(T_i\pi) = p_{1,i}(\pi). \tag{10}$$

For $j \neq i$, the map T_j preserves both $p_{1,i}$ and $p_{2,i}$. If $j < i$, it acts entirely to the left of r_i, leaves A_i fixed, and does not change the value set U_i. If $j > i$, all values moved by T_j are smaller than $r_j < r_i$. Among the affected positions to the left of r_i, their relative order is preserved. If these positions are nonempty, they contain every element of U_i below r_j, and hence the minimum of U_i; if they are empty, that minimum is fixed. Likewise, $\max U_i$ is fixed when it is at least $r_j + 1$, while otherwise all of U_i lies in the order-preservingly refilled positions. Thus the positions of both extrema, and hence their membership in A_i, are unchanged.

Finally, apply $T_1, T_2, \ldots, T_\ell$ in this order and call the result $T(\pi)$. This is a bijection on $\mathcal{A}_n$, with inverse obtained by applying the same involutions in reverse order. Equations (9) and (10), together with the preceding invariance, give

$$p_{2,i}(T(\pi)) = p_{1,i}(\pi) \quad (1 \le i \le \ell), \qquad \text{and hence} \qquad p_2(T(\pi)) = p_1(\pi).$$

Thus p_1 and p_2 are equidistributed on $\mathcal{A}_n$. $\square$

4 Concluding remarks

The computation distinguishes 203 distribution-equivalence classes of length 2 mesh patterns on alternating permutations, while the proofs above establish the claimed equivalences within each class. The examples further demonstrate that classifications over all permutations and over alternating permutations are not formally comparable: some bijections establishing equivalences for all permutations do not preserve alternation, whereas several equivalences in the latter setting arise from local peak-valley constraints that are invisible in the unrestricted setting.

As a direction for further research, we propose classifying distribution equivalence for other well-studied classes of permutations. Determining the distributions explicitly, including for the equivalence classes considered in this paper, is also an interesting avenue for future research. We conclude with two examples illustrating the form that such enumerative results can take. For $i \in \{10, 16\}$, let $p^{(i)}$ be the first representative displayed in the proof of Theorem 3.10 or Theorem 3.16, respectively, and put

$$P_{i,n}(q) := \sum_{\pi \in \mathcal{A}_n} q^{p^{(i)}(\pi)}.$$

For Class 10, the proof of Theorem 3.10 shows that $p^{(10)}$ takes only the values 0 and 1, and that $p^{(10)}(\pi) = 1$ precisely when $\pi = A\,1\,B$, where $B \neq \varnothing$ and every element of A is larger than every element of B. If $|A| = m$, then A must contain the m largest elements of $\{2, \ldots, n\}$. The alternating types of A and B are determined by their lengths, and their relative orders can be chosen independently in $E_m E_{n-m-1}$ ways. Therefore, for $n \geq 2$,

$$P_{10,n}(q) = |\mathcal{A}_n| + (q-1)\sum_{m=0}^{n-2} E_m E_{n-m-1}.$$

For Class 16, Equation (8), derived in the proof of Theorem 3.16, gives $P_{16,n}(q)$ for $n \geq 2$. Since all patterns within each class are equidistributed, these formulas apply to every representative of Classes 10 and 16. Finding comparably explicit formulas, or useful multivariate generating functions, for the remaining classes is a natural open problem.

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