

Graphs Capturing Alternations in Words

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A graph $G = (V, E)$ is representable if there exists a word W over the alphabet V such that letters x and y alternate in W if and only if $(x, y) \in E$ for each $x \neq y$. If W is k -uniform (each letter of W occurs exactly k times in it) then G is called k -representable. A graph is representable if and only if it is k -representable for some k [1].

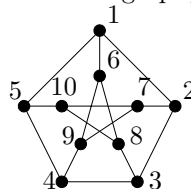
In this note, we introduce the applicability of representable graphs, and answer several open questions from [1].

Circular precedence constraints: Consider a scenario with n recurring tasks with requirements on the alternation of certain pairs of tasks. This captures typical situations in periodic scheduling, where there are recurring *precedence* requirements, e.g., “before each ignition, check the oil level”. When tasks occur only once, the pairwise requirements form precedence constraints, which are modeled by partial orders. When the directionality of the constraints is omitted, the resulting pairwise constraints form *comparability graphs*. We consider here graphs formed by pairwise alternation constraints

Execution sequences of recurring tasks can be viewed as words over an alphabet V , where V is the set of tasks. Thus, when tasks recur, the resulting alternation relationship forms a *representable* graph.

Proposition 1 ([1]). *Let $W = AB$ be a k -uniform word representing a graph G . Then the word $W' = BA$ also k -represents G .*

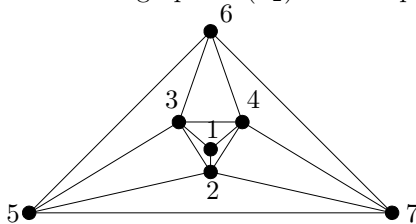
Representability of the Petersen graph: It was shown in [3] to be 3-representable: - 1, 3, 8, 7, 2, 9, 6, 10, 7, 4, 9, 3, 5, 4, 1, 2, 8, 3, 10, 7, 6, 8, 5, 10, 1, 9, 4, 5, 6, 2 - 1, 3, 4, 10, 5, 8, 6, 7, 9, 10, 2, 7, 3, 4, 1, 2, 8, 3, 5, 10, 6, 8, 1, 9, 7, 2, 6, 4, 9, 5 We can show that it is not 2-representable. Let W be a word 2-representing it. Some letter x must appear with the exactly three distinct letters between its two appearances. By symmetry and Prop. 1, $x = 1$ and W starts with 1. By symmetry and independence of 2,5,6, we can write $W = 12561W_16W_25W_32W_4$. To alternate with 6 but not to with 5, both W_1 and W_2 contain 8. To alternate with 2 but not with 5, both W_3 and W_4 contain 3. But then 8833 is a subsequence in W , so 8 and 3 are non-adjacent in the graph, a contradiction.



On forming non-representable graphs: The following open problem was posed in [1]: *Are there any non-representable graphs that do not satisfy the conditions of Theorem 1 below?*

Theorem 1. ([1]) *If G is representable, then for every $x \in V(G)$ the graph induced by $N(x)$ is permutationally representable, where $N(x)$ is the set of neighbors of x in G .*

We give a positive answer. A counterexample to the converse of Theorem 1 is given by the graph below called $\text{co}(T_2)$ in [4]. It is easy to check that the induced neighborhood of any node of the graph $\text{co}(T_2)$ is a comparability graph.



Theorem 2. *The graph $\text{co}(T_2)$ is non-representable.*

Proof. Assume that $\text{co}(T_2)$ is k -representable for some k and W is a word representing it. The vertices 1,2,3,4 form a clique; so, their appearances $1^i, 2^i, 3^i, 4^i$ in W must be in the same order for each $i = 1, 2, \dots, k$. By symmetry and Proposition 1 we may assume that the order is 1234. Now let I_1, I_2, \dots, I_k be the set of all $[2^i, 4^i]$ -intervals in W . Two cases are possible.

1. There is an interval I_j such that 7 belongs to it. Then since 2,4,7 form a clique, 7 must be inside each of the intervals I_1, I_2, \dots, I_k . But then 7 is adjacent to 1, a contradiction.
2. 7 does not belong to any of the intervals I_1, I_2, \dots, I_k . Again, since 7 is adjacent to 2 and 4, each pair of consecutive intervals I_j, I_{j+1} must be separated by a single 7. But then 7 is adjacent to 3, a contradiction.

The effect of graph operations: Finally, we observe that the following operation on a representable graph preserves representability: Replace any node with a comparability graph, connecting all the new nodes to the neighbors of the original node. I.e., replacing a node with a comparability graph *module*. On the other hand, several other operations on representable graphs do not necessarily result in a representable graph: Taking the complement, taking the line graph, or identifying cliques of size more than 1 from two representable graphs.

References

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4. <http://www.teo.informatik.uni-rostock.de/isgci/smallgraphs.html>