COUNTING INDEPENDENT SETS IN SOME CLASSES OF (ALMOST) REGULAR GRAPHS

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ABSTRACT. We enumerate the independent sets of several classes of regular and almost regular graphs and compute the corresponding generating functions. We also note the relations between these graphs and other combinatorial objects and, in some cases, construct the corresponding bijections.

1. INTRODUCTION

Let $I(G)$ denote the number of independent sets of a graph G. This number can be determined for some special classes of graphs (see [8] for a survey). For instance, $I(G)$ was studied for *grid graphs* (see [1]), *multipartite complete graphs*, *path-schemes* (see [7]), and *cyclic-schemes*. Some of these numbers are given by certain combinations of Fibonacci numbers, some others by Lucas numbers.

In this paper, we study four classes of graphs. To define these classes, we recall that a *line graph* $L(G)$ of a graph G is obtained by associating a vertex with each edge G and connecting two vertices with an edge if and only if the corresponding edges of G are

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adjacent. Also, recall that a *cycle graph* C_ℓ is a graph on ℓ nodes containing a single cycle through all nodes.

We now give the definitions of our classes.

Class 1: Let $G_\ell^1 = C_\ell$, the ℓ-cycle graph. We obtain G_ℓ^2 by superimposing the line graph $L(G_\ell^1)$ onto the graph G_ℓ^1, that is splitting each edge of G_ℓ^1 with the corresponding vertex of $L(G_\ell^1)$ and then adding the edges of $L(G_\ell^1)$. More generally, G_ℓ^n is obtained by superimposing $L^{n-1}(G_\ell^1)$ onto G_ℓ^{n-1}. For example, in Figure 1, we have the graphs G_3^4 and G_4^3, respectively, if one ignores the dashed edges. Clearly, all but n of the ℓ^n nodes of G_ℓ^n have degree 4, and we say that G_ℓ^n is an *almost* 4-regular graph.
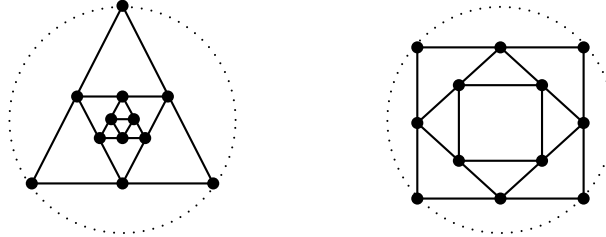

FIGURE 1. Examples of (almost) 4-regular graphs under consideration.

Class 2: The graph R_ℓ^n is obtained from G_ℓ^n by duplicating the edges of G_ℓ^1. For example, in Figure 1, the extra edges are the dashed edges, and by adding them we get the graphs R_3^4 and R_4^3 respectively. So, to get R_ℓ^n we add ℓ additional edges to G_ℓ^n, and it is easy to see that R_ℓ^n is a 4-regular graph.

Class 3: Let $K_\ell^1 = K_\ell$, a complete graph on ℓ nodes. Put the ℓ nodes of K_ℓ^1 on a circle and draw the remaining $(\ell - 1)! - \ell$ edges. Call the first ℓ edges *external* and the remaining edges, *internal*. Then construction of K_ℓ^n is similar to that of G_ℓ^n, except that:
  (1) The basis of the construction is now K_ℓ, rather than C_ℓ.
  (2) On each iteration i, the graph superimposed onto R_ℓ^{i-1} is not $L^i(K_\ell)$ but rather the complete graph on the nodes of $L^i(C_\ell)$, the line graph of C_ℓ formed by the external edges of K_ℓ^1.

   In Figure 2, we show how to construct K_4^3 from K_4^2 (the dashed edges should be ignored). In that figure, the external edges of respective complete graphs are in bold. We also remark that the internal edges do not intersect each other. Moreover, Figure 2 suggests a convenient way of representing K_4^3, where each node of the graph lies only on internal edges incident with that node. We achieve that by putting the nodes of the line graphs under consideration off the centers of the corresponding edges. Indeed, if we were using the centers of the external edges, K_4^3 would look as in Figure 3, which is misleading since, for example, the node a does not lie on the edge bc.

The graph K_ℓ^n is almost $(\ell+1)$-regular, since all but ℓ of its ℓ^n nodes have degree $\ell+1$. Moreover, it follows from our definitions that $K_3^n = G_3^n$.
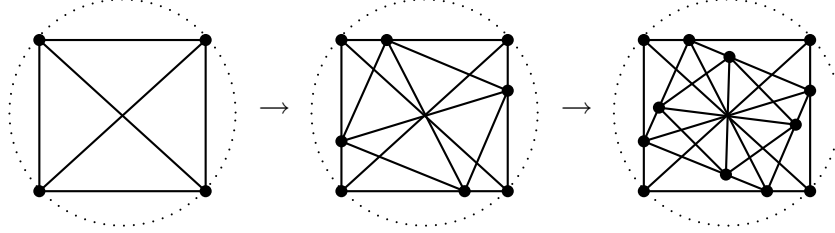

FIGURE 2. A way of constructing K_4^3 and P_4^3.

Class 4: The graph P_ℓ^n is obtained from the graph K_ℓ^n by duplicating the external edges in the graph K_ℓ^1. For example, in Figure 2, the extra edges are the dashed edges, and by adding them we construct the graph P_4^3 from K_4^3. So to get P_ℓ^n we add ℓ extra edges to K_ℓ^n, and it is easy to see that P_ℓ^n is an $(\ell+1)$-regular graph.
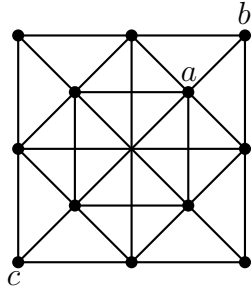

FIGURE 3. A bad presentation of K_4^3.

Let $g_\ell(n)$, $r_\ell(n)$, $k_\ell(n)$ and $p_\ell(n)$ denote the numbers of independent sets in the graphs G_ℓ^n, R_ℓ^n, K_ℓ^n and P_ℓ^n respectively. In our paper we study these numbers. In Section 3, we give an algorithm for calculating all these numbers. For the numbers $p_\ell(n)$, we provide an explicit generating function (see Theorem 3.4). However, in order to illustrate our approach to the problem, in Section 2 we consider $g_3(n)$ and find an explicit formula for it.

Our choice of the graphs to study was motivated by the so called *de Bruijn graphs*, which are defined as follows. A de Bruijn graph is a directed graph $\vec{G}_n = \vec{G}_n(V, E)$, where the set of vertices V is the set of all the words of length n in a finite alphabet A, and there is an arc from $v_i = (v_{i1}, \ldots, v_{in})$ to $v_j = (v_{j1}, \ldots, v_{jn})$ if

$$v_{i2} = v_{j1},\ v_{i3} = v_{j2}, \ldots, v_{in} = v_{j(n-1)},$$

that is when the words v_i and v_j overlap by $(n-1)$ letters.

The de Bruijn graphs were first introduced (for the alphabet $A = \{0,1\}$) by de Bruijn in 1944 for enumerating the number of code cycles. However, these graphs proved to be a useful tool for various problems related to the subject of combinatorics on words (e.g. see [2, 3, 5]). It is known that the graph $\vec{G}_n$ can be defined recursively as $\vec{G}_n = L(\vec{G}_{n-1})$. The authors were interested in studying other graphs defined recursively using the operation of taking line graphs (with natural bases), which could give interesting applications. Also, with our choice of graphs (G_ℓ^n and K_ℓ^n), it is natural to complete them to regular graphs (R_ℓ^n and P_ℓ^n) and study these graphs. It turns out that there are combinatorial interpretations (relations to other combinatorial objects) for the number of independent sets for some of our graphs, and we mention these relations in Sections 2 and 3. Moreover, we construct a direct bijection describing such a relation for P_4^n (see Proposition 3.5 and the discussion that follows).

2. The numbers $g_3(n)$.

Let us first find an explicit formula for $g_3(n)$. It is clear that for any independent set of the graph G_3^n, we can label a node of G_3^n 1 if this node is in the independent set, and label it 0 otherwise. Thus, our purpose is to count the number of triangles having either 0 or 1 in each node and such that no two adjacent nodes are both assigned 1s. We call such triangles *legal*.

In order to get a recursion for $g_3(n)$, we introduce three auxiliary parameters a_n, b_n, and c_n, which are the numbers of legal triangles that, up to rotation, have specific numbers in the nodes of the biggest triangle (see Figure 4). Since we consider only legal graphs, the 1s in the nodes of the biggest triangle induces 0s in certain nodes of a smaller triangle (this 0s are shown in Figure 4).
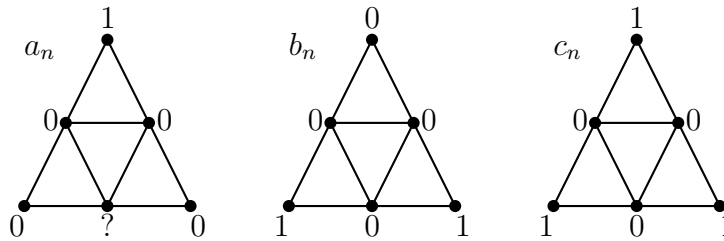

FIGURE 4. Auxiliary parameters a_n, b_n, and c_n.

Considering all the possibilities for the numbers of the biggest triangle, we have that

$$g_3(n) = g_3(n-1) + 3a_n + 3b_n + c_n,$$

where $g_3(n-1)$ corresponds to all 0s, and we have the multiple 3 twice because of possible rotations. Similarly, we get that $a_n = g_3(n-2) + a_{n-1}$, and $b_n = c_n = g_3(n-2)$. This leads to the recursion

(1) $$g_3(n) = 2g_3(n-1) + 6g_3(n-2) - 4g_3(n-3),$$

which, under the same initial conditions, is equivalent to the recursion

(2) $$g_3(n) = 4g_3(n-1) - 2g_3(n-2).$$

We define $g_3(0) = 1$, since we associate the graph G_3^0 with the empty graph, in which case there is only one independent set, the empty set. Thus,

$$g_3(n) = \frac{1}{2\sqrt{2}} \left((2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1} \right),$$

and the generating function for the numbers $g_3(n)$ is $1/(1 - 4x + 2x^2)$. The initial values for the numbers $g_3(n)$ are:

$$1, 4, 14, 48, 164, 560, 1912, 6528, 22288, 76096, \ldots.$$

Preceded by 0, the sequence $\{g_3(n)\}$ is the binomial transform of the Pell numbers

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

(see [11, A007070]). These numbers can also be interpreted as maximum bets in a poker game (also see [11, A007070]), where the first player bets 1 dollar into a pot and the ith player bets the amount of the $(i-1)$st player's bet plus the resulting amount of money in the pot. Then the number of dollars d_n in the pot after n bets is given by

$$d_n = 2(d_{n-1} + (d_{n-1} - d_{n-2})) = 4d_{n-1} - 2d_{n-2}, \quad d_0 = 1, d_1 = 4,$$

which yields $d_n = g_3(n)$.

We remark that it would be interesting to obtain recurrence (2) directly from the graph G_3^n, rather than via recurrence (1). Unfortunately, we were unable to do this.

3. An algorithm for calculating $g_\ell(n)$, $r_\ell(n)$, $k_\ell(n)$ and $p_\ell(n)$

In this section we present an algorithm for calculating $g_\ell(n)$, $r_\ell(n)$, $k_\ell(n)$ and $p_\ell(n)$ by using the transfer matrix method (see [10, Theorem 4.7.2]).

3.1. **An algorithm for calculating** $g_\ell(n)$**.** In this section, we use the transfer matrix method to obtain an information about the sequence of $g_\ell(n)$.

Similarly to Section 2, for any independent set of the graph G_ℓ^n, consider a labeling of G_ℓ^n, where the nodes of the independent set are labeled 1 and the remaining nodes are labeled 0. For a given graph G_ℓ^n, we define the *n-th level* of G_ℓ^n to be $G_\ell^n \setminus G_\ell^{n-1}$, which is isomorphic to G_ℓ^1. Thus, we may think of an independent set of the graph G_ℓ^n as assembled from elements chosen on each level, making sure that when we add a new level, we create no conflict with the previous level.

The collection $\mathcal{L}_\ell$ of possible level labelings is the set of all $(0,1)$ ℓ-vectors $\mathbf{v} = (v_1, \ldots, v_\ell)$. It will be convenient to define $v_{\ell+1} := v_1$. Then $\mathbf{v} = (v_1, \ldots, v_\ell)$ and $\mathbf{w} = (w_1, \ldots, w_\ell)$ in $\mathcal{L}_\ell$ are a possible consecutive pair of levels in an independent set of G_ℓ^n (with $\mathbf{w}$ following $\mathbf{v}$) if and only if

(3) $$v_i = v_{i+1} = 0 \text{ or } w_i = 0, \qquad \text{where } i = 1, 2, \ldots, \ell.$$

Thus, to obtain any independent set in the graph G_ℓ^n, we begin with a vector of $\mathcal{L}_\ell$, then keep adjoining each next vector $\mathbf{w} \in \mathcal{L}_\ell$ so that it satisfies (3) together with the previously chosen vector $\mathbf{v} \in \mathcal{L}_\ell$, until n vectors have been selected.

We define a matrix $G = G_\ell$, the transfer matrix of the problem, as follows. G is a $2^\ell \times 2^\ell$ matrix of 0s and 1s whose rows and columns are indexed by vectors of $\mathcal{L}_\ell$. The entry of G in position $(\mathbf{v}, \mathbf{w})$ is 1 if the ordered pair of vectors $(\mathbf{v}, \mathbf{w})$ satisfies (3), and is 0 otherwise. G depends only on ℓ, not on n. Hence, the number of independent sets of G_ℓ^n, $g_\ell(n)$, is the first entry of the vector $G^n \cdot (u_1, \ldots, u_{2^\ell})^T$, where $u_i = 1$ if the ith vector $\mathbf{v}$ in the collection $\mathcal{L}_\ell$ has no two consecutive 1s, even after wrapping, (i.e. $v_i + v_{i+1} \leq 1$ for all $i = 1, 2, \ldots, \ell$), and $u_i = 0$ otherwise. Hence,

$$g_\ell(n) = (1, 0, \ldots, 0) \cdot G^n \cdot (u_1, \ldots, u_{2^\ell})^T.$$

For instance, when $\ell = 3$, the possible level vectors are

$$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1),$$

except for the last level, where we only have

$$(0,0,0), (0,0,1), (0,1,0), (1,0,0).$$

If we index the rows and the columns of the transfer matrix G in this order, then we get

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The vector $(u_1, \ldots, u_{2^\ell})^T$ in this case is $(1,1,1,0,1,0,0,0)$. If we now find the first entry of the vector $(I - xG)^{-1} \cdot (1,1,1,0,1,0,0,0)^T$, where I is the unit matrix, then we get that the generating function for $g_3(n)$ is given by $1/(1 - 4x + 2x^2)$. We obtain the results for larger ℓ similarly.

Theorem 3.1. *The generating functions for the numbers $g_4(n)$, $g_5(n)$ and $g_6(n)$ are given, respectively, by*

$$\frac{1 + 4x - x^2 - 2x^3}{1 - 3x - 14x^2 + 15x^3 + 7x^4},$$

$$\frac{(1+x)(1 + 5x - 8x^2)}{1 - 5x - 30x^2 + 69x^3 + 31x^4 - 22x^5},$$

$$\frac{1 + 10x - 12x^2 - 50x^3 + 10x^4 + 20x^5 - 12x^6}{1 - 8x - 66x^2 + 280x^3 + 178x^4 - 532x^5 - 84x^6 + 108x^7}.$$

We remark that the algorithm for finding the generating function for $g_\ell(n)$ has been implemented in Maple, and yielded explicit results for $\ell \leq 6$.

3.2. **An algorithm for calculating $r_\ell(n)$.** In this section we use the transfer matrix method to obtain an information about the numbers $r_\ell(n)$. This case is similar to that of $g_\ell(n)$ with some small differences.

We partition R_ℓ^n into levels just as in the case of G_ℓ^n, so the n-th level of R_ℓ^n is $R_\ell^n \backslash R_\ell^{n-1}$.

The collection of possible levels $\mathcal{L}_\ell$ is the set of all ℓ-vectors $\mathbf{v}$ of 0s and 1s such that there no consecutive 1s in $\mathbf{v}$, that is, $v_i + v_{i+1} \neq 2$ (where we define $v_{\ell+1} := v_1$). Clearly, the set $\mathcal{L}_\ell$ contains exactly L_ℓ vectors where L_ℓ is the ℓth Lucas number. For instance, $\mathcal{L}_3$ contains the vectors $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$.

The condition that vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{L}_\ell$ are a possible consecutive pair of levels in an independent set of R_ℓ^n is given by (3) just as for G_ℓ^n. To obtain any independent set in the graph R_ℓ^n, we begin with a vector of $\mathcal{L}_\ell$, then keep adjoining each next vector $\mathbf{w} \in \mathcal{L}_\ell$ so that it satisfies (3) together with the previously chosen vector $\mathbf{v} \in \mathcal{L}_\ell$, until n vectors have been selected.

We define the transfer matrix of the problem $R = R_\ell$ in the same way as $G = G_\ell$ in subsection 3.1. Then R is an $L_\ell \times L_\ell$ matrix, and the number of independent sets of R_ℓ^n, $r_\ell(n)$, is the first entry of of the vector $R^n \cdot \mathbf{1}$ where $\mathbf{1} = (1, 1, \ldots, 1)$. Hence,

$$r_\ell(n) = (1, 0, \ldots, 0) \cdot R^n \cdot \mathbf{1}.$$

For instance, when $\ell = 3$, the possible level vectors in an independent set are

$$(0,0,0), (0,0,1), (0,1,0), (1,0,0).$$

If we indexed the rows and columns in this order, then the transfer matrix is

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

If we find the first entry of the vector $(I - xG)^{-1} \cdot \mathbf{1}$, we get that the generating function for $r_3(n)$ is given by $\frac{1+2x}{1-2x-2x^2}$. The initial values for the numbers $r_3(n)$ are

$$1, 4, 10, 28, 76, 208, 568, 1552, 4240, \ldots$$

This sequence appears as A026150 in [11].

Similarly to the case $\ell = 3$, we obtain the following results for $\ell = 4, 5, 6$.

Theorem 3.2. *The generating functions for the numbers $r_4(n)$, $r_5(n)$ and $r_6(n)$ are given, respectively, by*

$$\frac{1 + 4x - 4x^2}{1 - 3x - 4x^2 + 4x^3},$$

$$\frac{1 + 7x - 6x^2}{1 - 4x - 8x^2 + 6x^3},$$

$$\frac{1 + 12x - 24x^2 + 8x^4}{(1 - 8x + 4x^2 + 4x^3)(1 + 2x - 2x^2)}.$$

We remark that the algorithm for finding the generating function for $r_\ell(n)$ has been implemented in Maple, and yielded explicit results for $\ell \leq 6$.

3.3. **An algorithm for calculating $k_\ell(n)$.** In this section we use the transfer matrix method yet again to obtain information about the sequences $k_\ell(n)$. This case is also similar to that of $g_\ell(n)$, so we will only sketch it briefly.

We partition K_ℓ^n into levels just as in the case of G_ℓ^n, so the n-th level of K_ℓ^n is $K_\ell^n \backslash K_\ell^{n-1}$. The collection of possible levels $\mathcal{L}_\ell$ is the set of all ℓ-vectors $\mathbf{v}$ of 0s and 1s. Vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{L}_\ell$ are a possible consecutive pair of levels in an independent set of K_ℓ^n if they satisfy (3).

We define the transfer matrix of the problem, $K = K_\ell$, in the same way as G_ℓ. Then K is a $2^\ell \times 2^\ell$ matrix, and the number of independent sets of K_ℓ^n, $k_\ell(n)$, is the first entry of the vector $K^n \cdot (u_1, \ldots, u_{2^\ell})^T$ where $u_i = 1$ if the ith vector in the collection $\mathcal{L}_\ell$ contains at most one nonzero entry. Hence,

$$k_\ell(n) = (1, 0, \ldots, 0) \cdot K^n \cdot (u_1, \ldots, u_{2^\ell})^T.$$

For instance, when $\ell = 3$, the possible level vectors in an independent set are

$$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1).$$

If we index the rows and columns in this order, then the transfer matrix is

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we find the first entry of the vector $(I - xK)^{-1} \cdot (u_1, \ldots, u_{2^\ell})^T$, we get that the generating function for $k_3(n)$ is given by $\frac{1}{1-4x+2x^2}$. In particular, we get that $k_3(n) = g_3(n)$ which can also be seen directly from the definitions. Similarly, we have the following result.

Theorem 3.3. *The generating functions for the numbers $k_4(n)$, $k_5(n)$ and $k_6(n)$ are given, respectively, by*

$$\frac{1 + 2x + 3x^2}{1 - 3x - 14x^2 + 15x^3 + 7x^4},$$

$$\frac{1 + x + 12x^2 - 8x^3}{1 - 5x - 30x^2 + 69x^3 + 31x^4 - 22x^5},$$

$$\frac{1 - x + 38x^2 - 72x^3 - 8x^4 + 30x^5}{1 - 8x - 66x^2 + 280x^3 + 178x^4 - 532x^5 - 84x^6 + 108x^7}.$$

We remark that the algorithm for finding the generating function for $k_\ell(n)$ has been implemented in Maple, and yielded explicit results for $\ell \leq 6$.

3.4. **An algorithm for calculating $p_\ell(n)$.** In this section we use the transfer matrix method to obtain information about the sequences $p_\ell(n)$.

We partition P_ℓ^n into levels the same way as G_ℓ^n, so the n-th level of P_ℓ^n is $P_\ell^n \backslash P_\ell^{n-1}$. The collection of possible levels $\mathcal{L}_\ell$ is the set of all ℓ-vectors $\mathbf{v}$ of 0s and 1s. Vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{L}_\ell$ are a possible consecutive pair of levels in an independent set of P_ℓ^n if they satisfy (3).

The collection of possible levels $\mathcal{L}_\ell$ is the set of all ℓ-vectors $\mathbf{v} = (v_1, \ldots, v_\ell)$ of 0s and 1s such that $v_1 + \cdots + v_\ell \leq 1$. Clearly, the set $\mathcal{L}_\ell$ contains exactly $\ell + 1$ vectors which are $(0, \ldots, 0)$ and $(0, \ldots, 0, 1, 0, \ldots, 0)$. The condition that vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{L}_\ell$ are a possible consecutive pair of levels in an independent set of P_ℓ^n is given by (3).

We define the transfer matrix of the problem, $P = P_\ell$, in the same way as G_ℓ, R_ℓ and K_ℓ. We define a matrix $P = P_\ell$, the transfer matrix of the problem, as follows. P is an $(\ell + 1) \times (\ell + 1)$ matrix of 0s and 1s whose rows and columns are indexed by vectors of $\mathcal{L}_\ell$. Therefore, it is easy to see that

$$
P = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\
\vdots & & & & \vdots & & & & \vdots \\
1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0
\end{bmatrix}_{(\ell+1) \times (\ell+1)}.
$$

Theorem 3.4. *The generating function for $p_\ell(n)$ is given by*

$$
\sum_{n \geq 0} p_\ell(n) x^n = \frac{1 + 2x}{1 - (\ell - 1)x - 2x^2}.
$$

Proof. We want to find the first entry of the vector $(I - xP)^{-1} \cdot \mathbf{1}$ which means we must find the first row, say $(e_1, \ldots, e_{\ell+1})$, of the matrix $(I - xP)^{-1}$. By solving the system of equations

$$
(I - xP)^{-1} \cdot (e_1, \ldots, e_{\ell+1})^T = (1, 0, \ldots, 0)^T,
$$

we get that

$$
e_1 = \frac{1 - (\ell - 2)x}{1 - (\ell - 1)x - 2x^2} \text{ and } e_j = \frac{x}{1 - (\ell - 1)x - 2x^2} \text{ for } j \geq 2.
$$

Hence, the first entry of the vector $(I - xP)^{-1} \cdot \mathbf{1}$ is given by

$$
\sum_{i=1}^{\ell+1} e_i = \frac{1 + 2x}{1 - (\ell - 1)x - 2x^2}. \qquad \square
$$

For instance, when $\ell = 3$, the generating function for $p_3(n)$ is given by $\frac{1+2x}{1-2x-2x^2}$. One can see, in particular, that $p_3(n) = r_3(n)$, which follows directly from the definitions.

In the case $\ell = 4$ the initial values of the numbers $p_4(n)$ are

$$
1, 5, 17, 61, 217, 773, 2753, 9805, 34921, 124373, \ldots
$$

The same sequence turns out to appear in [6] (see [11, A007483]). Thus, the following proposition is true:

Proposition 3.5. *The number of independent sets in the graph P_4^n is equal to the number of (possibly empty) subsequences of the sequence $(1, 2, \ldots, 2n + 1)$ in which each odd member has an even neighbor.*

Here, the neighbors of an integer m are $m-1$ and $m+1$. In the case $n = 1$, the sequences appearing in the proposition are $\epsilon, 2, 23, 12, 123$, where ϵ is the empty sequence. In this case, we can find a direct bijection between the objects in Proposition 3.5 as described below.

We start by labeling the vertices of the (innermost) level n of P_4^n clockwise by 2, 23, 12, 123. The level $n - 1$ is labeled starting from the vertex immediately to the left of vertex labeled 2 as follows: $4, 45, [3]4, [3]45$ (the meaning of brackets will be discussed below). More generally, for $i < n$, given a level $n - i + 1$ labeled clockwise with $2i, 2i(2i + 1), [2i - 1]2i, [2i - 1]2i(2i + 1)$, we label the (next outer) level $n - i$ clockwise from the inside out with $2i + 2, (2i + 2)(2i + 3), [2i + 1](2i + 2), [2i + 1](2i + 2)(2i + 3)$ starting from the vertex immediately to the left of vertex labeled $2i$. (See Figure 5 for the case $n = 3$.) Each independent set has at most one vertex on each level. Now, given any independent set in P_4^n, we can write the labels of its nodes in increasing order and delete any integer $[2i - 1]$ in brackets if the sequence also contains $2i - 2$ or $2i - 1$ without brackets. Erasing all brackets now, if any, we obtain a sequence from Proposition 3.5.
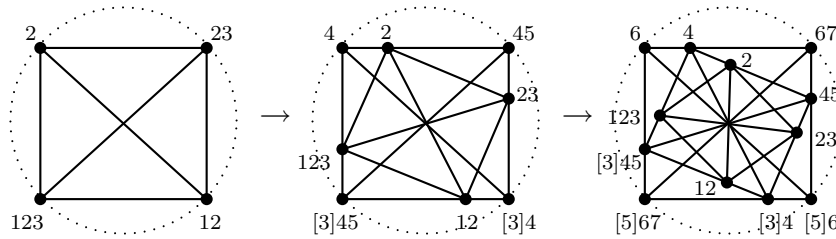

FIGURE 5. A labeling of P_4^3 by subsequences of Proposition 3.5.

Example 3.6. *The independent sets $\{45, [5]67\}, \{4, [5]67\}, \{12, [5]6\}, \{[3]4, 67\}$ correspond to the sequences $4567, 467, 1256, 3467$, respectively.*

For convenience, we will write down the set of nonadjacent labels at level $n - i$ for each label at level $n - i + 1$.

$$
\begin{aligned}
\text{(no label) } \epsilon &\mapsto (2i + 2), (2i + 2)(2i + 3), [2i + 1](2i + 2), [2i + 1](2i + 2)(2i + 3) \\
2i &\mapsto [2i + 1](2i + 2), [2i + 1](2i + 2)(2i + 3) \\
2i(2i + 1) &\mapsto (2i + 2), [2i + 1](2i + 2)(2i + 3) \\
[2i - 1]2i &\mapsto (2i + 2), (2i + 2)(2i + 3) \\
[2i - 1]2i(2i + 1) &\mapsto (2i + 2)(2i + 3), [2i + 1](2i + 2)
\end{aligned}
$$

(4)

Now it is not difficult to construct an independent set given a sequence of Proposition 3.5. We partition the sequence of integers from 1 to $2n + 1$ as follows:

$$123 \,|\, 45 \,|\, 67 \,|\ldots|\, 2n(2n + 1),$$

then choose the vertices of the independent set in the order of increasing labels using the rules (4). Notice that the label of the vertex at level $n - i + 1$ must contain $2i$.

Example 3.7.

$$4567 \mapsto 123|\underline{45}|\underline{67} \mapsto (\epsilon, 45, [5]67) \mapsto \{45, [5]67\}$$
$$467 \mapsto 123|\underline{45}|\underline{67} \mapsto (\epsilon, 4, [5]67) \mapsto \{45, [5]67\}$$
$$1256 \mapsto \underline{123}|45|\underline{67} \mapsto (12, \epsilon, [5]6) \mapsto \{12, [5]6\}$$

It can be shown that the two maps described above are inverses of each other based on the recursive structure of sequences under consideration.

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