# Introduction to Partially Ordered Patterns

Sergey Kitaev[*]

Reykjavík University

Ofanleiti 2

IS-103 Reykjavík, Iceland

*Dedicated to the 50th anniversary of my advisor, Einar Steingrímsson*

## Abstract

We review selected known results on partially ordered patterns (POPs) that include co-unimodal, multi- and shuffle patterns, peaks and valleys ((modified) maxima and minima) in permutations, the Horse permutations and others. We provide several new results on a class of POPs built on an arbitrary flat poset, obtaining, as corollaries, the bivariate generating function for the distribution of peaks (valleys) in permutations, links to Catalan, Narayana, and Pell numbers, as well as generalizations of a few results in the literature including the descent distribution. Moreover, we discuss a $q$-analogue for a result on non-overlapping segmented POPs. Finally, we suggest several open problems for further research.

**Keywords:** (partially ordered) pattern, non-overlapping occurrences, peak, valley, $q$-analogue, flat poset, co-unimodal pattern, bijection, (exponential, bivariate) generating function, distribution, Catalan numbers, Narayana numbers, Pell numbers

## 1  Introduction and background

An occurrence of a *pattern* $\tau$ in a permutation $\pi$ is defined as a subsequence in $\pi$ (of the same length as $\tau$) whose letters are in the same relative order as those in $\tau$. For example, the permutation 31425 has three occurrences of the pattern 1-2-3, namely the subsequences 345, 145, and 125. *Generalized permutation patterns* (*GPs*) being introduced in [2] allow the requirement that some adjacent letters in a pattern must also be adjacent in the permutation. We indicate this requirement by removing a dash in the corresponding place. Say, if pattern 2-31 occurs in a permutation $\pi$, then the letters in $\pi$ that correspond to 3 and 1 are adjacent. For example, the permutation 516423 has only one occurrence of the pattern 2-31, namely the subword

---

564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563. Placing "[" on the left (resp., "]" on the right) next to a pattern $p$ means the requirement that $p$ must begin (resp., end) from the leftmost (resp., rightmost) letter. For example, the permutation 32415 contains two occurrences of the pattern [2-13, namely the subwords 324 and 315 and no occurrences of the pattern 3-2-1].

A further generalization of the GPs is *partially ordered patterns* (*POPs*), where the letters of a pattern form a partially ordered set (poset), and an occurrence of such a pattern in a permutation is a linear extension of the corresponding poset in the order suggested by the pattern (we also pay attention to eventual dashes and brackets). For instance, if we have a poset on three elements labeled by $1'$, 1, and 2, in which the only relation is $1 < 2$ (see Figure 1), then in an occurrence of $p = 1'$-12 in a permutation $\pi$ the letter corresponding to the $1'$ in $p$ can be either larger or smaller than the letters corresponding to 12. Thus, the permutation 31254 has three occurrences of $p$, namely 3-12, 3-25, and 1-25.
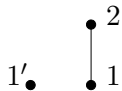


Figure 1: A poset on three elements with the only relation $1 < 2$.

Let $\mathcal{S}_n(p_1, \ldots, p_k)$ denote the set of $n$-permutations avoiding simultaneously each of the patterns $p_1, \ldots, p_k$.

The POPs were introduced in [16][1] as an auxiliary tool to study the maximum number of non-overlapping occurrences of *segmented* GPs (*SGPs*), also known as *consecutive* GPs, that is, the GPs, occurrences of which in permutations form contiguous subwords (there are no dashes). However, the most useful property of POPs known so far is their ability to "encode" certain sets of GPs which provides a convenient notation for those sets and often gives an idea how to treat them. For example, the original proof of the fact that $|\mathcal{S}_n(123, 132, 213)| = \binom{n}{\lfloor n/2 \rfloor}$ took 3 pages ([15]); on the other hand, if one notices that $|\mathcal{S}_n(123, 132, 213)| = |\mathcal{S}_n(11'2)|$, where the letters 1, $1'$, and 2 came from the same poset as above, then the result is easy to see. Indeed, we may use the property that the letters in odd and even positions of a "good" permutation do not affect each other because of the form of $11'2$. Thus we choose the letters in odd positions in $\binom{n}{\lfloor n/2 \rfloor}$ ways, and we must arrange them in decreasing order. We then must arrange the letters in even positions in decreasing order too.

The POPs can be used to encode certain combinatorial objects by restricted permutations. Examples of that are Propositions 10 and 13, as well as several other propositions in [5]. Such encoding is interesting from the point of view of finding bijections, but it also may have applications for

---

[1]The POPs in this paper are the same as the POGPs in [16], which is an abbreviation for Partially Ordered Generalized Patterns.

enumerating certain statistics. The idea is to encode a set of objects under consideration as a set of permutations satisfying certain restrictions (given by certain POPs); under appropriate encodings, this allows us to transfer the interesting statistics from the original set to the set of permutations, where they are easy to handle. For an illustration of how encodings by POPs can be used, see [19, Thm. 2.4] which deals with POPs in *compositions* rather than in permutations, though, but the approach remains the same.

As a matter of fact, some POPs appeared in the literature before they were actually introduced. Thus the notion of a POP allows us to collect under one roof (to provide a uniform notation for) several combinatorial structures such as *peaks*, *valleys*, *modified maxima* and *modified minima* in permutations, *Horse permutations* and *p-descents* in permutations discussed in Section 2.

This paper is organized as follows. Section 2 reviews selected results in the literature related to POPs; Section 3 provides a complete solution for SPOPs built on a *flat poset*[2] without repeated letters. In particular, as a corollary to a more general result, we provide the generating function for the distribution of peaks (valleys) in permutations, which seems to be a new result, or at least one the author could not find in the literature (it looks like only a continued fraction expansion of the generating function for the distribution of peaks is known). Section 4 gives a $q$-analogue for a result on non-overlapping patterns ([17, Thm. 16]). Finally, in Section 5, we state several open problems on POPs.

In what follows we need the following notations. Let $\sigma$ and $\tau$ be two POPs of length greater than 0. We write $\sigma < \tau$ to indicate that any letter of $\sigma$ is less than any letter of $\tau$. We write $\sigma <> \tau$ when no letter in $\sigma$ is comparable with any letter in $\tau$. Also, *SPOP* abbreviates Segmented POP.

A *left-to-right minimum* of a permutation $\pi$ is an element $a_i$ such that $a_i < a_j$ for every $j < i$. Analogously we define *right-to-left minimum*, *right-to-left maximum*, and *left-to-right maximum*. If $\pi = a_1 a_2 \cdots a_n \in \mathcal{S}_n$, then the *reverse* of $\pi$ is $\pi^r := a_n \cdots a_2 a_1$, and the *complement* of $\pi$ is a permutation $\pi^c$ such that $\pi_i^c = n + 1 - a_i$, where $i \in [n] = \{1, \ldots, n\}$. We call $\pi^r$, $\pi^c$, and $(\pi^r)^c = (\pi^c)^r$ *trivial bijections*. The *GF* (*EGF*; *BGF*) denotes the (*exponential*; *bivariate*) *generating function*.

# 2 Review of selected results on POPs

In this section we review several results in the literature related to POPs.

## 2.1 Co-unimodal patterns

For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{S}_n$, the *inversion index*, $\text{inv}(\pi)$, is the number of ordered pairs $(i, j)$ such that $1 \leq i < j \leq n$ and $\pi_i > \pi_j$. The

---

[2]The concept of a "flat poset" is used in theoretical computer science [1] to denote posets with one element being less than any other element (there are no other relations between the elements). See Figure 5 for the shape of such poset.

major index, $\mathrm{maj}(\pi)$, is the sum of all $i$ such that $\pi_i > \pi_{i+1}$. Suppose $\sigma$ is a SPOP and

$$\mathrm{place}_\sigma(\pi) = \{i \mid \pi \text{ has an occurrence of } \sigma \text{ starting at } \pi_i\}.$$

Let $\mathrm{maj}_\sigma(\pi)$ be the sum of the elements of $\mathrm{place}_\sigma(\pi)$.
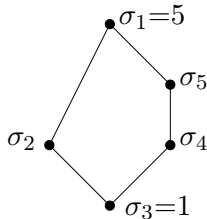


Figure 2: A poset for co-unimodal pattern in the case $j = 3$ and $k = 5$.

If $\sigma$ is *co-unimodal*, meaning that $k = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \le j \le k$ (see Figure 2 for a corresponding poset in the case $j = 3$ and $k = 5$), then the following formula holds [4]:

$$\sum_{\pi \in \mathcal{S}_n} t^{\mathrm{maj}_\sigma(\pi^{-1})} q^{\mathrm{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{maj}_\sigma(\pi^{-1})} q^{\mathrm{inv}(\pi)}.$$

If $k = 2$ we deal with usual descents, thus a co-unimodal pattern can be viewed as a generalization of the notion of a descent. This may be a reason why a co-unimodal pattern $p$ is called *p-descent* in [4]. Also, setting $t = 1$ we get a well-known result by MacMahon on equidistribution of maj and inv.

## 2.2 Peaks and valleys in permutations

A permutation $\pi$ has exactly $k$ *peaks* (resp., *valleys*), also known as *maxima* (resp., *minima*), if $|\{j \mid \pi_j > \max\{\pi_{j-1}, \pi_{j+1}\}\}| = k$ (resp., $|\{j \mid \pi_j < \min\{\pi_{j-1}, \pi_{j+1}\}\}| = k$). Thus, an occurrence of a peak in a permutation is an occurrence of the SPOP $1'21''$, where relations in the poset are $1' < 2$ and $1'' < 2$. Similarly, occurrences of valleys correspond to occurrences of the SPOP $2'12''$, where $2' > 1$ and $2'' > 1$. See Figure 3 for the posets corresponding to the peaks and valleys. So, any research done on the peak (or valley) statistics can be regarded as research on (S)POPs (e.g., see [26]).
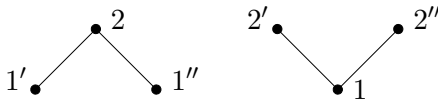


Figure 3: Posets corresponding to peaks and valleys.

Also, results related to *modified maxima* and *modified minima* can be viewed as results on SPOPs. For a permutation $\sigma_1 \ldots \sigma_n$ we say that $\sigma_i$ is a *modified maximum* if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ and a *modified minimum* if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$, for $i = 1, \ldots, n$, where $\sigma_0 = \sigma_{n+1} = 0$. Indeed, we can

view a pattern $p$ as a function from the set of all symmetric groups $\cup_{n\geq 0}\mathcal{S}_n$ to the set of natural numbers such that $p(\pi)$ is the number of occurrences of $p$ in $\pi$, where $\pi$ is a permutation. Thus, studying the distribution of modified maxima (resp., minima) is the same as studying the function $ab]+1'21''+[dc$ (resp., $ba]+2'12''+[cd)$ where $a<b$, $c<d$ and the other relations between the patterns' letters are taken from Figure 3. Also, recall that placing "[" (resp., "]") next to a pattern $p$ means the requirement that $p$ must begin (resp., end) with the leftmost (resp., rightmost) letter.

A specific result in this direction is problem 3.3.46(c) on page 195 in [13]: We say that $\sigma_i$ is a *double rise* (resp., *double fall*) if $\sigma_{i-1}<\sigma_i<\sigma_{i+1}$ (resp., $\sigma_{i-1}>\sigma_i>\sigma_{i+1}$); The number of permutations in $\mathcal{S}_n$ with $i_1$ modified minima, $i_2$ modified maxima, $i_3$ double rises, and $i_4$ double falls is

$$\left[u_1^{i_1}u_2^{i_2-1}u_3^{i_3}u_4^{i_4}\frac{x^n}{n!}\right]\frac{e^{\alpha_2 x}-e^{\alpha_1}x}{\alpha_2 e^{\alpha_1 x}-\alpha_1 e^{\alpha_2 x}}$$

where $\alpha_1\alpha_2=u_1u_2$, $\alpha_1+\alpha_2=u_3+u_4$.

In Corollary 23 we obtain explicit generating function for the distribution of peaks (valleys) in permutations. This result is an analogue to a result in [9] where the circular case of permutations is considered, that is, when the first letter of a permutation is though to be to the right of the last letter in the permutation. In [9] it is shown that if $M(n,k)$ denotes the number of circular permutations in $\mathcal{S}_n$ having $k$ maxima, then

$$\sum_{n\geq 1}\sum_{k\geq 0}M(n,k)y^k\frac{x^n}{n!}=\frac{zx(1-z\tanh xz)}{z-\tanh xz}$$

where $z=\sqrt{1-y}$.

## 2.3 Patterns containing □-symbol

In [14] the authors study simultaneous avoidance of the patterns 1-3-2 and 1□23. A permutation $\pi$ avoids 1□23 if there is no $\pi_i<\pi_j<\pi_{j+1}$ with $i<j-1$. Thus the □ symbol has the same meaning as "-" except for □ does not allow the letters separated by it to be adjacent in an occurrence of the corresponding pattern. In the POP-terminology, 1□23 is the pattern 1-1'-23, or 1-1'23, or 11'-23, where 1' is incomparable with the letters 1, 2, and 3 which, in turn, are ordered naturally: $1<2<3$. The permutations avoiding 1-3-2 and 1□23 are called *Horse permutations*. The reason for the name came from the fact that these permutations are in one to one correspondence with *Horse paths*, which are the lattice paths from (0,0) to $(n,n)$ containing the steps $(0,1)$, $(1,1)$, $(2,1)$, and $(1,2)$ and not passing the line $y=x$. According to [14], the generating function for the horse permutations is

$$\frac{1-x-\sqrt{1-2x-3x^2-4x^3}}{2x^2(1+x)}.$$

Moreover, in [14] the generating functions for Horse permutations avoiding, or containing (exactly) once, certain patterns are given.

In [10], patterns of the form $x$-$y\square z$ are studied, where $xyz \in \mathcal{S}_3$. Such a pattern can be written in the POP-notation as, for example, $x$-$y$-$a$-$z$ where $a$ is not comparable to $x$, $y$, and $z$. A bijection between permutations avoiding the pattern 1-2$\square$3, or 2-1$\square$3, and the set of *odd-dissection convex polygons* is given. Moreover, generating functions for permutations avoiding 1-3$\square$2 and certain additional patterns are obtained in [10].

## 2.4  A pattern of the form $\sigma$-$m$-$\tau$

Let $\sigma$ and $\tau$ be two SGPs (the results below work for SPOPs as well). We consider the POP $\alpha = \sigma$-$m$-$\tau$ with $m > \sigma$, $m > \tau$, and $\sigma <> \tau$, that is, each letter of $\sigma$ is incomparable with any letter of $\tau$ and $m$ is the largest letter in $\alpha$. The POP $\alpha$ is an instance of so called *shuffle patterns* (see [16, Sec 4]).

**Theorem 1.** ([16, Thm. 16]) *Let $A(x)$, $B(x)$ and $C(x)$ be the EGF for the number of permutations that avoid $\sigma$, $\tau$ and $\alpha$ respectively. Then $C(x)$ is the solution to the following differential equation with $C(0) = 1$:*

$$C'(x) = (A(x) + B(x))C(x) - A(x)B(x).$$

If $\tau$ is the empty word then $B(x) = 0$ and we get the following result for segmented GPs:

**Corollary 2.** ([16, Thm. 13],[20]) *Let $\alpha = \sigma$-$m$, where $\sigma$ is a SGP on $[k-1]$. Let $A(x)$ (resp., $C(x)$) be the EGF for the number of permutations that avoid $\sigma$ (resp., $\alpha$). Then $C(x) = e^{F(x,A)}$, where $F(x, A) = \int_0^x A(y)\ dy$.*

**Example 1.** ([16, Ex 15]) Suppose $\alpha = 12$-$3$. Here $\sigma = 12$, whence $A(x) = e^x$, since there is only one permutation that avoids $\sigma$. So

$$C(x) = e^{F(x,\exp)} = e^{e^x - 1}.$$

We get [6, Prop. 4] since $C(x)$ is the EGF for the Bell numbers.

**Corollary 3.** ([16, Cor. 19]) *Let $\alpha = \sigma$-$m$-$\tau$ is as described above. We consider the pattern $\varphi(\alpha) = \varphi_1(\sigma)$-$m$-$\varphi_2(\tau)$, where $\varphi_1$ and $\varphi_2$ are any trivial bijections. Then $|\mathcal{S}_n(\alpha)| = |\mathcal{S}_n(\varphi(\alpha))|$.*

## 2.5  Multi-patterns

Suppose $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is a set of segmented GPs and $p = \sigma_1$-$\sigma_2$-$\cdots$-$\sigma_k$ where each letter of $\sigma_i$ is incomparable with any letter of $\sigma_j$ whenever $i \neq j$ ($\sigma_i <> \sigma_j$). We call such POPs *multi-patterns*. Clearly, the Hasse diagram for such a pattern is $k$ disjoint chains similar to that in Figure 4.

**Theorem 4.** ([16, Thm. 23 and Cor. 24]) *The number of permutations avoiding the pattern $p = \sigma_1$-$\sigma_2$-$\cdots$-$\sigma_k$ is equal to that avoiding a multi-pattern obtained from $p$ by an arbitrary permutation of $\sigma_i$'s as well as by applying to $\sigma_i$'s any of trivial bijections.*
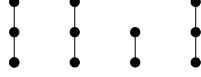
Figure 4: A poset corresponding to a multi-pattern.

The following theorem is the basis for calculating the number of permutations that avoid a multi-pattern.

**Theorem 5.** ([16, Thm. 28]) *Let $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ be a multi-pattern and let $A_i(x)$ be the EGF for the number of permutations that avoid $\sigma_i$. Then the EGF $A(x)$ for the number of permutations that avoid $p$ is*

$$A(x) = \sum_{i=1}^{k} A_i(x) \prod_{j=1}^{i-1} ((x-1)A_j(x) + 1).$$

**Corollary 6.** ([16, Cor. 26]) *Let $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ be a multi-pattern, where $|\sigma_i| = 2$ for all $i$. That is, each $\sigma_i$ is either 12 or 21. Then the EGF for the number of permutations that avoid $p$ is given by*

$$A(x) = \frac{1 - (1 + (x-1)e^x)^k}{1-x}.$$

**Remark 7.** Although the results in Theorems 4 and 5 are stated in [16] for $\sigma_i$'s which are SGPs, one can see that the same arguments work for $\sigma_i$'s which are SPOPs. Thus we have a generalization of these theorems.

## 2.6 Non-overlapping patterns – an application of POPs

This subsection deals additionally with occurrences of patterns in words. The letters $1, 2, 1', 2'$ appearing in the examples below are ordered as in Figure 7 to be found on page 20.

Theorem 5 and its counterpart in the case of words [18, Thm. 4.3] and [18, Cor. 4.4], as well as Remark 7 applied for these results, give an interesting application of the multi-patterns in finding a certain statistic, namely the *maximum number of non-overlapping occurrences of a SPOP* in permutations and words. For instance, the maximum number of non-overlapping occurrences of the SPOP $11'2$ in the permutation 621394785 is 2, and this is given by the occurrences 213 and 478, or the occurrences 139 and 478.

Theorem 8 generalizes [16, Thm. 32] and [18, Thm. 5.1].

**Theorem 8.** ([17, Thm. 16]) *Let $p$ be a SPOP and $B(x)$ (resp., $B(x; k)$) is the EGF (resp., GF) for the number of permutations (resp., words over $[k]$) avoiding $p$. Let $D(x, y) = \sum_\pi y^{N(\pi)} \frac{x^{|\pi|}}{|\pi|!}$ and $D(x, y; k) = \sum_{n \geq 0} \sum_{w \in [k]^n} y^{N(w)} x^n$*

where $N(s)$ is the maximum number of non-overlapping occurrences of $p$ in $s$. Then $D(x, y)$ and $D(x, y; k)$ are given by

$$\frac{B(x)}{1 - y(1 + (x - 1)B(x))} \quad and \quad \frac{B(x; k)}{1 - y(1 + (kx - 1)B(x; k))}.$$

The following examples are corollaries to Theorem 8.

**Example 2.** ([17, Ex 1]) If we consider the SPOP $11'$ then clearly $B(x) = 1 + x$ and $B(x; k) = 1 + kx$. Hence,

$$D(x, y) = \frac{1 + x}{1 - yx^2} = \sum_{i \geq 0} (x^{2i} + x^{2i+1}) y^i,$$

and

$$D(x, y; k) = \frac{1 + kx}{1 - y(kx)^2} = \sum_{i \geq 0} ((kx)^{2i} + (kx)^{2i+1}) y^i.$$

**Example 3.** ([17, Ex 2]) For permutations, the distribution of the maximum number of non-overlapping occurrences of the SPOP $122'1'$ is given by

$$D(x, y) = \frac{\frac{1}{2} + \frac{1}{4} \tan x (1 + e^{2x} + 2e^x \sin x) + \frac{1}{2} e^x \cos x}{1 - y(1 + (x - 1)(\frac{1}{2} + \frac{1}{4} \tan x (1 + e^{2x} + 2e^x \sin x) + \frac{1}{2} e^x \cos x))}.$$

## 2.7 Segmented patterns of length four

In this subsection we provide the known results related to SPOPs of length four. Corollaries 17 and 22 in subsection 3.1 give extra results in this direction. In subsection 5.3 we provide unsolved cases with initial values for the number of the restricted permutations. In this subsection, $A(x) = \sum_{n \geq 0} A_n x^n / n!$ is the EGF for the number of permutations in question. The patterns in the subsection are built on the poset from Figure 7 and the letter $1''$ is not comparable to any other letter.

**Theorem 9.** ([16, Thm. 30]) *For the SPOP $122'1'$, we have that*

$$A(x) = \frac{1}{2} + \frac{1}{4} \tan x (1 + e^{2x} + 2e^x \sin x) + \frac{1}{2} e^x \cos x.$$

**Proposition 10.** ([17, Prop. 8,9]) *There are $\binom{n-1}{\lfloor (n-1)/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor}$ permutations in $S_n$ that avoid the SPOP $12'21'$. The $(n+1)$-permutations avoiding $12'21'$ are in one-to-one correspondence with different walks of $n$ steps between lattice points, each in a direction N, S, E or W, starting from the origin and remaining in the positive quadrant.*

**Proposition 11.** ([17, Prop. 4,5,6]) *For the SPOP $11'1''2$, one has*

$$A_n = \frac{n!}{\lfloor n/3 \rfloor! \lfloor (n+1)/3 \rfloor! \lfloor (n+2)/3 \rfloor!},$$

*and for the SPOP $11'21''$ and $n \geq 1$, we have $A_n = n \cdot \binom{n-1}{\lfloor (n-1)/2 \rfloor}$. Moreover, for the SPOPs $1'1''12$ and $1'121''$, we have $A_0 = A_1 = 1$, and, for $n \geq 2$, $A_n = n(n-1)$.*

8

**Proposition 12.** ([17, Prop. 7]) *For the SPOP* $1231'$, *we have*

$$A(x) = xe^{x/2}\left(\cos\frac{\sqrt{3}x}{2} - \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}x}{2}\right)^{-1} + 1,$$

*and for the SPOPs* $1321'$ *and* $2131'$, *we have*

$$A(x) = x(1 - \int_0^x e^{-t^2/2}\ dt)^{-1} + 1.$$

We end up this subsection with a result on multi-avoidance of SPOPs that has a combinatorial interpretation.

**Proposition 13.** ([5, Prop. 2.1,2.2]) *There are* $2\binom{n}{\lfloor n/2\rfloor}$ *n-permutations avoiding the SPOPs* $11'22'$ *and* $22'11'$ *simultaneously. For* $n \geq 3$, *there is a bijection between such n-permutations and the set of all* $(n+1)$-*step walks on the x-axis with the steps* $a = (1,0)$ *and* $\bar{a} = (-1,0)$ *starting from the origin but not returning to it.*

## 3   Patterns built on flat posets

In this section, we consider flat posets built on $k + 1$ elements $a, a_1, \ldots, a_k$ with the only relations $a < a_i$ for all $i$. A Hasse diagram for the flat poset is in Figure 5.
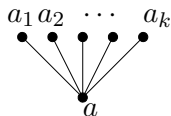


Figure 5: A flat poset.

### 3.1   Avoidance and distribution of the patterns

The following proposition generalizes [6, Prop. 6]. Indeed, letting $k = 2$ in the proposition we deal with involutions and permutations avoiding 1-23 and 1-32.

**Proposition 14.** *The permutations in* $\mathcal{S}_n$ *having cycles of length at most* $k$ *are in one-to-one correspondence with permutations in* $\mathcal{S}_n$ *that avoid* $a\text{-}a_1\cdots a_k$.

*Proof.* We construct a bijection in a similar to [6, Prop. 6] way.

Let $\pi \in \mathcal{S}_n$ be a permutation with cycles of length at most $k$. A standard form for writing $\pi$ in cycle notation is requiring that

   (1)  Each cycle is written with its least element first;

   (2)  The cycles are written in decreasing order of their least element.

Define $\hat{\pi}$ to be the permutation obtained from $\pi$ by writing it in standard form and erasing the parentheses separating the cycles. The permutation $\hat{\pi}$ avoids $a\text{-}a_1 \cdots a_k$. Indeed, the distance between two left-to-right minima (the number of letters between them) in $\hat{\pi}$ does not exceed $k - 1$ because of the restriction on the cycle lengths. Thus if $\hat{\pi}$ contains $a\text{-}a_1 \cdots a_k$ then among the letters of $\hat{\pi}$ corresponding to $a_1 \cdots a_k$ there is at least one left-to-right minimum, say $m$, and the letter in $\hat{\pi}$ corresponding to $a$ must be less than $m$. This contradicts the definition of a left-to-right minimum.

Conversely, if $\hat{\pi}$ is an $a\text{-}a_1 \cdots a_k$-avoiding permutation then any two of its consecutive left-to-right minima are at distance not exceeding $k - 1$ from each other, since otherwise we have an occurrence of $a\text{-}a_1 \cdots a_k$ starting at a left-to-right minimum preceding a factor of length at least $k$ that does not contain other left-to-right minima. The left-to-right minima of $\hat{\pi}$ define cycles of $\pi$. $\square$

**Corollary 15.** *The EGF for the number of permutations avoiding $a\text{-}a_1 \cdots a_k$ is given by* $\exp(\sum_{i=1}^{k} x^i/i)$.

*Proof.* According to Proposition 14 we only need to find the EGF $p(x) = \sum_{n \geq 0} p_n x^n/n!$ for the number of permutations with cycles of length at most $k$, which is known (see, e.g., [12]), but we rederive it here.

Suppose $\pi$ is an $n$-permutation with cycles of length at most $k$ and 1 occurs in a cycle $C$. If $i$ is the number of neighbors of 1 in $C$ then $0 \leq i \leq k - 1$ and there are $\binom{n-1}{i} i!$ possibilities for choosing such $C$. Thus

$$p_n = \sum_{i=0}^{k-1} \binom{n-1}{i} i! p_{n-i-1}$$

which after summing over all $n \geq 1$ gives

$$p'(x) = (1 + x + \cdots + x^{k-1}) p(x)$$

and therefore the claim is true since $P(0) = 1$. $\square$

**Proposition 16.** *One has $\mathcal{S}_n(a\text{-}a_1 \cdots a_k) = \mathcal{S}_n(aa_1 \cdots a_k)$, and thus the EGF for the number of permutations avoiding $aa_1 \cdots a_k$ is* $\exp(\sum_{i=1}^{k} x^i/i)$.

*Proof.* Clearly $\mathcal{S}_n(a\text{-}a_1 \cdots a_k) \subseteq \mathcal{S}_n(aa_1 \cdots a_k)$. Suppose now that $\pi \in \mathcal{S}_n(aa_1 \cdots a_k)$ and $\pi$ contains an occurrence of $a\text{-}a_1 \cdots a_k$, say $\pi_i \pi_j \pi_{j+1} \cdots \pi_{j+k-1}$ where $i + 1 < j$. We will get a contradiction which will show that $\mathcal{S}_n(aa_1 \cdots a_k) \subseteq \mathcal{S}_n(a\text{-}a_1 \cdots a_k)$.

One can assume that $j - i$ is minimal out of all occurrences of $a\text{-}a_1 \cdots a_k$ in $\pi$. If $\pi_{j-1} < \pi_i$ then $\pi_{j-1} \pi_j \pi_{j+1} \cdots \pi_{j+k-1}$ is an occurrence of $aa_1 \cdots a_k$, a contradiction to $\pi \in \mathcal{S}_n(aa_1 \cdots a_k)$; otherwise, $\pi_i \pi_{j-1} \pi_j \cdots \pi_{j+k-2}$ is an occurrence of $a\text{-}a_1 \cdots a_k$, a contradiction to $j - i$ being minimal. $\square$

**Corollary 17.** *The EGF for the number of permutations avoiding $aa_1 a_2 a_3$ is given by* $\exp(x + x^2/2 + x^3/3)$.

**Theorem 18.** (Distribution of $aa_1a_2\cdots a_k$) Let

$$P := P(x, y) = \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} y^{e(\pi)} x^n / n!$$

be the BGF on permutations, where $e(\pi)$ is the number of occurrences of the SPOP $p = aa_1a_2\cdots a_k$ in $\pi$. Then $P$ is the solution of

$$\frac{\partial P}{\partial x} = yP^2 + \frac{(1-y)(1-x^k)}{1-x}P \tag{1}$$

with the initial condition $P(0, y) = 1$.

*Proof.* Suppose $\pi = \pi' 1 \pi''$ is a permutation. Then

$$e(\pi) = \begin{cases} e(\pi') + e(\pi'') + 1 & \text{if } |\pi''| \geq k, \\ e(\pi') & \text{if } |\pi''| < k \end{cases}$$

since an occurrence of $p$ cannot start at $\pi'$ and end not in $\pi'$; also when $\pi''$ is of length at least $k$ it contributes one extra occurrence of $p$ starting at 1.

Suppose $P_{<k} := P_{<k}(x, y) = \sum_{n=0}^{k-1} \sum_{\pi \in \mathcal{S}_n} y^{e(\pi)} x^n / n! = \sum_{n \geq 0} x^n = \frac{1-x^k}{1-x}$. Readers familiar with the symbolic method can now see that

$$P' = P(y(P - P_{<k}) + P_{<k})$$

with the initial condition $P(0, y) = 1$ and the desired is easy to get by plugging in $P_{<k}$ and rewriting the equation.

The rest of the proof is dedicated to a brief explanation of the symbolic method (see [11] for more details) and applying it to our case. In our presentation we follow [8].

There is a direct correspondence between set-theoretic operations on combinatorial classes and algebraic operations on EGFs. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be classes of labeled combinatorial objects, and $A(x)$, $B(x)$, and $C(x)$ be their EGFs respectively. Then if $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ is the union of disjoint copies then $A(x) = B(x) + C(x)$; if $\mathcal{A} = \mathcal{B} \star \mathcal{C}$ is the labeled product, that is, the usual Cartesian product enriched with the relabeling operation, then $A(x) = B(x)C(x)$; if $\mathcal{A} = \mathcal{B}^\square \star \mathcal{C}$ is the box product, that is, the subset of $\mathcal{B} \star \mathcal{C}$ formed by those pairs in which the smallest label lies in the $\mathcal{B}$ component, then $A(x) = \int_0^x (\frac{d}{dt} B(t)) \cdot C(t) dt$. The same holds if we have the BGFs instead of EGFs.

Let $\mathcal{P}$ be the class of all permutations and $\mathcal{P}_{<k}$ is the class of permutations of length less than $k$. With some abuse of notation, we introduce the parameter $y$ in the equation for classes meaning that it will be placed there when we write the corresponding differential equations for the BGFs. With this notation and using the property of $e(\pi)$, we can write

$$\mathcal{P} = \{\epsilon\} + \{x\}^\square \star \mathcal{P} \star [y(\mathcal{P} - \mathcal{P}_{<k}) + \mathcal{P}_{<k}]$$

where $\epsilon$ is the empty permutation. We differentiate the corresponding equation for BFGs to get the desired result. $\qquad\square$

Note, that if $y = 0$ in Theorem 18 then the function in Corollary 15, due to Proposition 16, is supposed to be the solution to (1), which is true. If $k = 1$ in Theorem 18, then as the solution to (1) we get nothing else but the distribution of descents in permutations: $(1-y)(e^{(y-1)x} - y)^{-1}$. Thus Theorem 18 can be thought as a generalization of the result on the descent distribution.

The following theorem generalizes Theorem 18. Indeed, Theorem 18 is obtained from Theorem 19 by plugging in $\ell = 0$ and observing that obviously $aa_1 \cdots a_k$ and $a_1 \cdots a_k a$ are equidistributed.

**Theorem 19.** (Distribution of $a_1 a_2 \cdots a_k a a_{k+1} a_{k+2} \cdots a_{k+\ell}$) *Let*

$$P := P(x,y) = \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n} y^{e(\pi)} x^n / n!$$

*be the BGF of permutations where $e(\pi)$ is the number of occurrences of the SPOP $p = a_1 a_2 \cdots a_k a a_{k+1} a_{k+2} \cdots a_{k+\ell}$ in $\pi$. Then $P$ is the solution of*

$$\frac{\partial P}{\partial x} = y \left( P - \frac{1-x^k}{1-x} \right) \left( P - \frac{1-x^\ell}{1-x} \right) + \frac{2 - x^k - x^\ell}{1-x} P - \frac{1 - x^k - x^\ell + x^{k+\ell}}{(1-x)^2}. \tag{2}$$

*with the initial condition $P(0, y) = 1$.*

*Proof.* A proof is straightforward applying the technique introduced in the proof of Theorem 18. We use the same notation and adjusted steps of that proof without explanations.

Suppose $\pi = \pi' 1 \pi''$ is a permutation. Then

$$e(\pi) = \begin{cases} e(\pi') + e(\pi'') + 1 & \text{if } |\pi'| \geq k \text{ and } |\pi''| \geq \ell , \\ e(\pi') + e(\pi'') & \text{otherwise.} \end{cases}$$

One can now see that $\mathcal{P}$ is equal to

$$\{\epsilon\} + \{x\}^\square \star [y(\mathcal{P} - \mathcal{P}_{<k}) \star (\mathcal{P} - \mathcal{P}_{<\ell}) + (P - \mathcal{P}_{<k}) \star \mathcal{P}_{<\ell} + \mathcal{P}_{<k} \star (\mathcal{P} - \mathcal{P}_{<\ell}) + \mathcal{P}_{<k} \star \mathcal{P}_{<\ell}]$$

and the rest is obtained by rewriting in terms of BGFs and differentiating. $\square$

If $y = 0$ in Theorem 19 then we get the following corollary:

**Corollary 20.** *The EGF $A(x) = \sum_{n \geq 0} A_n x^n / n!$ for the number of permutations avoiding the SPOP $p = a_1 a_2 \cdots a_k a a_{k+1} a_{k+2} \cdots a_{k+\ell}$ satisfies the following differential equation with the initial condition $A(0) = 1$:*

$$A'(x) = \frac{2 - x^k - x^\ell}{1-x} A(x) - \frac{1 - x^k - x^\ell + x^{k+\ell}}{(1-x)^2}.$$

The following corollaries to Corollary 20 are obtained by plugging in $k = \ell = 1$ and $k = 1$ and $\ell = 2$ respectively.

**Corollary 21.** ([15]) The EGF for the number of permutations avoiding $a_1aa_2$ is $(\exp(2x) + 1)/2$ and thus $|\mathcal{S}_n(a_1aa_2)| = 2^{n-1}$.

**Corollary 22.** *The EGF for the number of permutations avoiding $a_1aa_2a_3$ is*

$$1 + \sqrt{\frac{\pi}{2}}(erf(\frac{1}{\sqrt{2}}x + \sqrt{2}) - erf(\sqrt{2}))e^{\frac{1}{2}x(x+4)+2}$$

*where $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$ is the error function.*

If $k = 1$ and $\ell = 1$ then our pattern $a_1aa_2$ is nothing else but the valley statistic. In [24] a recursive formula for the generating function of permutations with exactly $k$ valleys is obtained, which however does not seem to allow (at least easily) finding the corresponding BGF. As a corollary to Theorem 19 we get the following BGF by solving (2) for $k = 1$ and $\ell = 1$:

**Corollary 23.** *The BGF for the distribution of peaks (valleys) in permutations is given by*

$$1 - \frac{1}{y} + \frac{1}{y}\sqrt{y-1} \cdot \tan\left(x\sqrt{y-1} + \arctan\left(\frac{1}{\sqrt{y-1}}\right)\right).$$

Expanding the BGF in Corollary 23 we can get, for example, the sequences A000431, A000487, and A000517 appearing in [25] for the number of permutations with exactly one, two, and three valleys respectively. Note, that we have already obtained the number of *valleyless* permutations in Corollary 21. The valleyless permutations were studied in [24].

## 3.2 Distribution of the patterns with additional restrictions

The results from this subsection are in a similar direction as that in the papers [3], [21], [22], and several other papers, where the authors study 1-3-2-avoiding permutations with respect to avoidance/count of other patterns. Such a study not only gives interesting enumerative results, but also provides a number of applications (see [3]).

To state the theorem below, we define $P_k = \sum_{n=0}^{k-1} \frac{1}{n+1}\binom{2n}{n}x^n$. That is, $P_k$ is the sum of initial $k$ terms in the expansion of the generating function $\frac{1-\sqrt{1-4x}}{2x}$ of the Catalan numbers.

**Theorem 24.** (Distribution of $a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ on $\mathcal{S}_n$(2-1-3)) *Let*

$$P := P(x,y) = \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(\text{2-1-3})} y^{e(\pi)}x^n$$

*be the BGF of 2-1-3-avoiding permutations where $e(\pi)$ is the number of occurrences of the SPOP $p = a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ in $\pi$. Then $P$ is given by*

$$\frac{1 - x(1-y)(P_k + P_\ell) - \sqrt{(x(1-y)(P_k + P_\ell) - 1)^2 - 4xy(x(y-1)P_kP_\ell + 1)}}{2xy}.$$

*Proof.* Let $\pi = \pi_1 1 \pi_2 \in \mathcal{S}_n(\text{2-1-3})$. Then each letter in $\pi_1$ must be greater than any letter in $\pi_2$, where both $\pi_1$ and $\pi_2$ must necessarily be 2-1-3-avoiding. Conversely, every permutation of this form is clearly 2-1-3-avoiding.

It is easy to see that $e(\pi) = e(\pi_1) + e(\pi_2) + \delta_{|\pi_1|,|\pi_2|}$, where

$$
\delta_{|\pi_1|,|\pi_2|} = \begin{cases} 1 & \text{if } |\pi_1| \geq k \text{ and } |\pi_2| \geq \ell \ , \\ 0 & \text{otherwise.} \end{cases}
$$

Using the symbolic method we get that, in terms of GFs,

$$
P = 1 + x(y(P - P_k)(P - P_\ell) + P_k \cdot P + P \cdot P_\ell - P_k \cdot P_\ell)
$$

where 1 corresponds to the empty permutation, and we subtracted $P_k \cdot P_\ell$ since the permutations corresponding to this term are counted twice, namely in $P_k \cdot P$ and in $P \cdot P_\ell$.

To get the desired we solve the equation above for $P$. $\qquad\square$

We now discuss several corollaries to Theorem 24. Note that letting $y = 1$ we obtain the GF for the Catalan numbers. Also, letting $y = 0$ in the expansion of $P$, we obtain the GF for the number of permutations avoiding simultaneously the patterns 2-1-3 and $a_1 a_2 \cdots a_k a a_{k+1} a_{k+2} \cdots a_{k+\ell}$.

If $k = 1$ and $\ell = 0$ in Theorem 24, then $P_k = 1$ and $P_\ell = 0$, and we obtain the distribution of descents in 2-1-3-avoiding permutations. This distribution gives the *triangle of Narayana numbers* (see [25, A001263] for more details).

If $k = \ell = 1$ in Theorem 24 then we deal with avoiding the pattern 2-1-3 and counting occurrences of the pattern 312, since any occurrence of $a_1 a a_2$ in a legal permutation must be an occurrence of 312 and vice versa. Thus the BGF of 2-1-3-avoiding permutations with a prescribed number of occurrences of 312 is given by

$$
\frac{1 - 2x(1 - y) - \sqrt{4(1 - y)x^2 + 1 - 4x}}{2xy}.
$$

Reading off the coefficients of the terms involving only $x$ in the expansion of the function above, we can see that the number of $n$-permutations avoiding simultaneously the patterns 2-1-3 and 312 is $2^{n-1}$, which is known and is easy to see directly from the structure of such permutations.

Reading off the coefficients of the terms involving $y$ to the power 1 we see that the number of $n$-permutations avoiding 2-1-3 and having exactly one occurrence of the pattern 312 is given by $(n - 1)(n - 2)2^{n-4}$. The corresponding sequence appears as [25, A001788] and gives an interesting fact which we state as Proposition 25. We give a combinatorial proof of that fact.

**Proposition 25.** *There is a bijection between 2-dimensional faces in the $(n+1)$-dimensional hypercube and 2-1-3-avoiding $(n+2)$-permutations with exactly one occurrence of the pattern 312.*

*Proof.* Recall that a node in a hypercube is at level $i$ if the binary vector corresponding to it contains $i$ 1's.

A 2-dimensional face in $(n + 1)$-dimensional hypercube can be specified by choosing two positions in an $(n+1)$-binary vector and fixing the remaining entries of the vector to be 0 or 1 (in $2^{n-1}$ ways). Indeed, any 2-dimensional face in a hypercube is a 4-cycle having two nodes at the same, say $i$-th, level, one node at the $(i + 1)$-st level and one node at the $(i - 1)$-st level. Moreover, the binary vectors corresponding to the nodes from the $i$-th level must differ only in two coordinates and thus one of the vectors has 1 and 0 in these coordinates whereas the second vector has 0 and 1 there. So, the number of 2-dimensional faces in the $(n+1)$-dimensional hypercube is given by $\binom{n+1}{2}2^{n-1}$ which is the same as the number of the $(n + 2)$-permutations under consideration (we refer to such permutations as "good permutations").

We now describe the structure of the good permutations. Suppose $\pi = \pi_1 1 \pi_2$ is a good permutation. Clearly, to avoid 2-1-3, any letter of $\pi_1$ must be greater than any letter of $\pi_2$. If $\pi_1$ and $\pi_2$ are non-empty, then the unique occurrence of the pattern 312 involves 1 in $\pi$ and both $\pi_1$ and $\pi_2$ must avoid simultaneously 2-1-3 and 312. The permutations avoiding both 2-1-3 and 312 have one peak, that is, the elements to the right (resp., left) of the largest element must be in decreasing (resp., increasing) order. If $\pi_1$ (resp., $\pi_2$) is empty, then $\pi_2$ (resp., $\pi_1$) is a good permutation and we use induction on length to describe the structure of $\pi$.

Given a 2-dimensional face defined by a binary vector

$$\mathbf{v} = a_1 \cdots a_i x a_{i+2} \cdots a_j y a_{j+2} \cdots a_{n+1}$$

with chosen positions $i + 1$ and $j + 1$ filled by $x$ and $y$ (if $y$ is next to $x$ then $j = i + 1$; if $x$ is the leftmost element then $i = 0$; if $y$ is the rightmost element then $j = n$). Based on the structure considerations above, we describe a procedure to find a good $(n+2)$-permutation corresponding to $\mathbf{v}$. We read $\mathbf{v}$ from left to right and place $1, 2, \ldots, n + 2$, one by one, into our permutation $\pi = \pi_1 \pi_2 \cdots \pi_{n+2}$ which we think of as being initially $n + 2$ empty slots. If we write, say, $\pi'_k$ then we mean that the $k$-th slot of $\pi$ is filled.

We start filling $\pi$ by reading $a_k$, $k = 1, 2, \ldots, i$: if $a_k = 0$, place $k$ into the leftmost empty slot of $\pi$; place $k$ into the rightmost empty slot otherwise. Suppose that as the result of filling the first $i$ elements we get $\pi'_1 \cdots \pi'_t \pi_{t+1} \cdots \pi_{n+t-i+2} \pi'_{n+t-i+3} \cdots \pi'_{n+2}$. Set $\pi_{t+j-i+1} = i + 1$. Note that currently we have the word $\pi'_1 \cdots \pi'_t A(i+1) B \pi'_{n+t-i+3} \cdots \pi'_{n+2}$, where $A$ and $B$ consist of empty slots, $|A| = j - i \geq 1$ and $|B| = n - j + 1 \geq 1$. In what follows, any element to be filled in $A$ is greater than any element to be filled in $B$, and thus the element $i + 1$ is involved in an occurrence of the pattern 312. This occurrence will be the only one in the permutation.

We fill in $B$ by reading $a_k$, $k = j + 2, \ldots, n + 1$ and placing the elements $(i + 2), \ldots, (n - j + i + 1)$, one by one, as follows: if $a_k = 0$, place the current element into the leftmost empty slot of $B$; place the current element insto the rightmost empty slot otherwise. We place $(n - j + i + 2)$ in the remaining empty slot of $B$. Fill in the remaining elements, one by one in

increasing order, into $A$ by reading $a_k$, $k = i + 2, \ldots, j$ in the way similar to that when proceeding with $B$. In particular, $n + 2$ will be placed in the remaining empty slot of $A$.

For example, the face $110x0y01$ corresponds to the permutation $389457621$, where $A$ is filled by $89$ and $B$ by $576$.

Our map is obviously injective and the converse to it is easy to see. $\qquad \square$

If $k = 1$ and $\ell = 2$ in Theorem 24 then we deal with avoiding the pattern 2-1-3 and counting occurrences of the pattern $a_1 a a_2 a_3$. In particular, one can see that the number of permutations avoiding simultaneously 2-1-3 and $a_1 a a_2 a_3$ is given by the *Pell numbers* $p(n)$ defined as $p(n) = 2p(n-1) + p(n-2)$ for $n > 1$; $p(0) = 0$ and $p(1) = 1$. The Pell numbers appear as [25, A000129], where one can find objects related to our restricted permutations.

# 4   $q$-analogues for non-overlapping SPOPs

The purpose of this section is to prove Theorem 29 which is a $q$-analogue of [17, Thm. 16]. In fact, the formulation of Theorem 29 is similar to that of the $q$-analogue of [16, Thm. 32] obtained in [23]. Moreover, to prove Theorem 29 one can use the same arguments as those in [23] involving rather complicated considerations based on symmetric functions, but we choose a simpler proof that is similar to proving [16, Thm. 32] in [16].

We fix some notations. Let $p$ be a segmented POP (SPOP) and $A^p_{n,k}$ be the number of $n$-permutations avoiding $p$ and having $k$ inversions. As usually, $[n]_q = q^0 + \cdots + q^{n-1}$, $[n]_q! = [n]_q \cdots [1]_q$, $\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{[n]_q!}{[i]_q![n-i]_q!}$, and, as above, $\mathrm{inv}(\pi)$ denotes the number of inversions in a permutation $\pi$. We set $A^p_n(q) = \sum_{\pi \text{ avoids } p} q^{\mathrm{inv}(\pi)}$. Moreover,

$$A^p_q(x) = \sum_{n,k} A^p_{n,k} q^k \frac{x^n}{[n]_q!} = \sum_n A^p_n(q) \frac{x^n}{[n]_q!} = \sum_{\pi \text{ avoids } p} q^{\mathrm{inv}(\pi)} \frac{x^{|\pi|}}{[|\pi|]_q!}.$$

All the definitions above are similar in case of permutations that *quasi-avoid* $p$, indicated by $B$ rather than $A$, namely, those permutations that have exactly one occurrence of $p$ and this occurrence consists of the $|p|$ rightmost letters in the permutations.

**Lemma 26.** (A $q$-analogue of [16, Prop. 4] that is valid for POPs) We have $B^p_q(x) = (x - 1)A^p_q(x) + 1$.

*Proof.* If we consider all $(n-1)$-permutations avoiding $p$ (the number of those, if we register inversions, is $A^p_{n-1}(q)$) and all possible extensions of these permutations to the $n$-permutations by writing one more letter to the right; then the number of obtained permutations, with inversions registered, is $(1 + q + \cdots + q^{n-1})A_{n-1}(q) = [n]_q A_{n-1}(q)$, where, for instance, $q^{n-1}$ in the sum corresponds to having 1 in the rightmost position. Obviously, the

set of these permutations is a disjoint union of the set of all $n$-permutations that avoid $p$ and the set of all $n$-permutations that quasi-avoid $p$. Thus, $B_n^p(q) = [n]_q A_{n-1}^p(q) - A_n^p(q)$. Multiplying both sides of the last equality by $x^n/[n]_q!$ and summing over all $n$ gives the desired result. $\square$

**Lemma 27.** (A $q$-analogue of [16, Thm. 25] that is valid for POPs) Let $P = p\text{-}\sigma$ be a POP, where $\sigma$ is an arbitrary POP built on the alphabet that is incomparable to that involved in a SPOP $p$. Then

$$A_q^P(x) = A_q^p(x) + A_q^\sigma(x)B_q^p(x).$$

*Proof.* If a permutation $\pi$ avoids $p$ then it avoids $P$. Otherwise we find the leftmost occurrence of $p$ in $\pi$. We assume that this occurrence consists of the $|p|$ rightmost letters among the $i$ leftmost letters of $\pi$. So the subword of $\pi$ beginning at the $(i+1)$st letter must avoid $\sigma$. From this, using independence between the first $i$ letters of $\pi$ and the remain letters, we conclude

$$A_n^P(q) = A_n^p(q) + \sum_{i=|\sigma|}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q B_i^p(q)A_{n-i}^\sigma(q).$$

Observe that one can change the lower bound in the sum above to 0, because $B_i^p(q) = 0$ for $i = 0, 1, \ldots, |p| - 1$. Multiplying both sides by $x^n/[n]_q!$ and summing over all $n$ we get the desired. $\square$

**Theorem 28.** (A $q$-analogue of [16, Thm. 28] that is valid for POPs) Let $p = p_1\text{-}\cdots\text{-}p_k$ be a multi-pattern ($p_i$s are SPOPs, and letters of $p_i$ and $p_j$ are incomparable for $i \neq j$). Then

$$A_q^p(x) = \sum_{i=1}^{k} A_q^{p_i}(x) \prod_{j=1}^{i-1} B_q^{p_j}(x) = \sum_{i=1}^{k} A_q^{p_i}(x) \prod_{j=1}^{i-1} ((x-1)A_q^{p_j}(x) + 1).$$

*Proof.* The first equality follows from lemma 27 by induction on $k$, and the second equality is then given by lemma 26. $\square$

**Theorem 29.** (A $q$-analogue of [17, Thm. 16]) If $N_p(\pi)$ denotes the maximum number of non-overlapping occurrences of a SPOP $p$ in $\pi$, then

$$\sum_{\pi} y^{N(\pi)} q^{inv(\pi)} \frac{x^{|\pi|}}{|\pi|!} = \frac{A_q^p(x)}{1 - yB_q^p(x)} = \frac{A_q^p(x)}{1 - y((x-1)A_q^p(x) + 1)}.$$

*Proof.* We fix $k$ and consider the multi-pattern $P_k = p\text{-}\cdots\text{-}p$ with $k$ copies of $p$. A permutation avoiding $P_k$ has at most $k-1$ non-overlapping occurrences of $p$. From Theorem 28,

$$A_q^{P_{k+1}}(x) - A_q^{P_k}(x) = A_q^p(x)(B_q^p(x))^k,$$

which is a bivariate generating function for the number of permutations with exactly $k$ non-overlapping occurrences of $p$ and with registered inversions. The result is now follows from summing over all $k$ and applying lemma 26. $\square$

# 5  Some open problems on POPs

We know very little on avoiding, and almost nothing on the distribution of, POPs. There are a lot of posets and different classes of posets, which provides enormous possibilities for further research. In particular, a natural step would be to extend/generalize results in the literature related to GPs to that related to POPs in the manner Proposition 14 and Theorem 8 are obtained. In this section, we state just few problems on POPs that might be interesting to solve.

## 5.1  Alternating patterns

A permutation $\pi_1 \pi_2 \ldots \pi_n$ is *alternating* (resp., *reverse alternating*) if $\pi_1 > \pi_2 < \pi_3 > \cdots$ (resp., $\pi_1 < \pi_2 > \pi_3 < \cdots$). It is well known that the EGF for the number of (reverse) alternating permutations is $\tan x + \sec x$.

We say that a permutation is a *k-non-alternating* (resp., *k-non-reverse-alternating*) if it does not contain $k$ consecutive letters that form an (resp., reverse) alternating permutation. Using the complement, one can see that the numbers of $k$-non-alternating and $k$-non-reverse-alternating $n$-permutations are the same.

**Problem 1.** Enumerate $k$-non-alternating $n$-permutations. (For $k = 4$ and $n \geq 4$ the numbers of "good" $n$-permutations are 19, 70, 331, 1863, 11637, 81110, ...; for $k = 5$ and $n \geq 5$ we have the sequence 104, 528, 3296, 23168, 179712,...)

**Problem 2.** Enumerate $n$-permutations that are both $k$-non-alternating and $k$-non-reverse-alternating. (For $k = 4$ and $n \geq 4$ we have the sequence 14, 52, 204, 1010, 5466, 34090,...; for $k = 5$ and $n \geq 5$ we have 24, 88, 458, 2716, 17808, 135182,...)

To generalize the problems above, we define a *k-alternating* (resp., *k-reverse-alternating*) pattern to be one that forms a (resp., reverse) alternating permutation of length $k$. Clearly, a $k$-alternating (resp., $k$-reverse-alternating) segmented pattern is a SPOP, where the corresponding poset is built on $k$ elements $a_1, \ldots, a_k$ with the relations $a_1 > a_2 < a_3 > \cdots$ (resp., $a_1 < a_2 > a_3 < \cdots$) (see Figure 6 for the case $k = 5$).
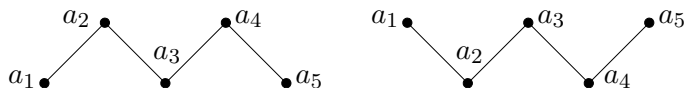


Figure 6: Posets for the 5-reverse-alternating and 5-alternating patterns.

Note that an occurrence of a descent in a permutation is an occurrence of a 2-alternating pattern. Thus we have yet another generalization of the notion of a descent beyond that discussed in subsection 2.1. Moreover, such

patterns generalize the patterns associated with peaks (valleys) in permutations, which gives a motivation to study them.

The number of descents in a permutation $\pi$ is denoted by $des(\pi)$. *Eulerian numbers* $A(n,k)$ count permutations in the symmetric group $\mathcal{S}_n$ with $k$ descents and they are the coefficients of the *Eulerian polynomials* $A_n(t)$ defined by $A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{1+des(\pi)}$. The Eulerian polynomials satisfy the identity

$$\sum_{k \geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

For more properties of the Eulerian polynomials see [7].

A natural generalization of the polynomials $A_n(t)$ is given by considering $k$-alternating patterns instead of descents in the definition of the polynomials. Let us call such new polynomials $B_n^k(t)$. From definitions, $A_n(t) = B_n^2(t)$.

**Problem 3.** Study the properties of the polynomials $B_n^2(t)$ and find the distribution for $k$-alternating patterns, that is, find an explicit formula for $B_n^2(t)$ or, if possible, coefficients of $B_n^2(t)$.

**Problem 4.** Find joint distribution for $k$-alternating and $k$-reverse-alternating patterns.

Note that there are many other (segmented) patterns that can be built on posets similar to those in Figure 6. For example, one could consider the pattern $a_1 a_3 a_5 a_2 a_4$ built on the poset to the left in Figure 6. To study other than alternating patterns built on such posets might be also an interesting direction to explore.

## 5.2 Co-unimodal patterns

Recall from subsection 2.1 that a SPOP $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$ is co-unimodal if $k = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \leq j \leq k$. We extend the concept of co-unimodal pattern to that of *free co-unimodal pattern* by removing the restriction "$k =$" in the definition. Note that co-unimodal patterns impose weaker restrictions on permutations than free co-unimodal patterns do.

**Problem 5.** How many of $n$-permutations avoid a co-unimodal pattern of length $k$. (For $k = 4$ and $j = 2$ (resp., $j = 3$) see the record for the pattern *utxv* (resp., *spor*) in table 1.)

**Problem 6.** How many of $n$-permutations avoid a free co-unimodal pattern of length $k$. (For $k = 4$, because of the complement, $j = 2$ and $j = 3$ give the same number of $n$-permutations avoiding them; see the record for the pattern *ijkm* in table 1.)

**Problem 7.** Find the distribution of a co-unimodal pattern of length $k$.

**Problem 8.** Find the distribution of a free co-unimodal pattern of length $k$.

**Problem 9.** Find the number of $n$-permutations avoiding simultaneously two or more of (free) co-unimodal patterns. We provide some numerical data in case $k = 4$. Suppose $F_2$ and $F_3$ are the free co-unimodal patterns corresponding to $j = 2$ and $j = 3$ respectively; also, $U_2$ and $U_3$ are the co-unimodal patterns corresponding to $j = 2$ and $j = 3$ respectively. The initial values for the number of $n$-permutations, $n \geq 4$, avoiding a pair of the patterns are as follows: $(F_2,F_3)$ – 18, 66, 252, 1176, 5768, 34216; $(F_2,U_3)$ – 19, 75, 330, 1753, 10319, 70011; $(F_3,U_2)$ – 20, 81, 372, 1981, 11866, 80043; $(U_2,U_3)$ – 21, 91, 462, 2718, 18181, 136491.

**Problem 10.** Find the joint distribution of two or more (free) co-unimodal patterns.

## 5.3 Remaining cases of SPOPs of length four

In table 1, we record few initial values for the number of $n$-permutations in some of unsolved cases of avoidance of SPOPs of length four, $n \geq 1$. In the table we record patterns having at least one pair of incomparable letters (see Figure 7 for the corresponding poset), although there are unsolved cases when all elements are comparable (we have a chain in the Hasse diagram). We refer to [8] for information on unsolved segmented GPs of length four. Table 1 is also an extended version of the corresponding table in [17].
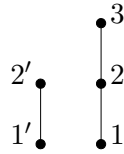


Figure 7: Poset from which some patterns in table 1 are built.

Other 4-SPOPs that were not considered can be built on the posets in Figure 8. For example, for the second poset there are three SPOPs to consider that are non-equivalent up to trivial bijections: $egfh$, $efgh$, and $fegh$ (see table 1 for corresponding sequences).
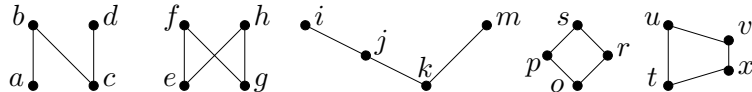


Figure 8: Five posets to build 4-SPOPs that were not considered.

Notice that the leftmost poset in Figure 8 can be used to build the 4-reverse-alternating pattern $abcd$, as well as the 4-alternating pattern $dcba$, whereas the third (resp., forth, fifth) poset in Figure 8 can be used to build

| | |
|---|---|
| $11'22'$ | 1, 2, 6, 18, 70, 300, 1435, 7910, 47376,... |
| $121'2'$ | 1, 2, 6, 18, 61, 281, 1541, 8920, 57924,... |
| $11'2'2$ | 1, 2, 6, 18, 71, 322, 1665, 9789, 64327,... |
| $12'1'2$ | 1, 2, 6, 18, 61, 272, 1410, 8048, 51550,... |
| $121'3$ | 1, 2, 6, 20, 83, 411, 2290, 14588, 104448,... |
| $131'2$ | 1, 2, 6, 20, 81, 390, 2161, 13678, 96983,... |
| $231'1$ | 1, 2, 6, 20, 83, 402, 2245, 14192, 100650,... |
| $abcd$ | 1, 2, 6, 19, 70, 331, 1863, 11637, 81110,... |
| $utxv$ | 1, 2, 6, 23, 110, 630, 4210, 32150, 276210,... |
| $spor$ | 1, 2, 6, 22, 100, 540, 3388, 24248, 195048,... |
| $ijkm$ | 1, 2, 6, 21, 90, 450, 2619, 17334, 129114,... |
| $egfh$ | 1, 2, 6, 20, 84, 412, 2300, 14676, 104536,... |
| $efgh$ | 1, 2, 6, 20, 80, 404, 2368, 15488, 114480,... |
| $fegh$ | 1, 2, 6, 20, 80, 360, 1888, 11168, 75168,... |

Table 1: The initial values for the number of $n$-permutations avoiding 4-SPOPs in a few of unsolved cases, $n \geq 1$. See Figures 7 and 8 for the corresponding poset.

free co-unimodal (resp., co-unimodal) pattern(s) of length 4, namely $ijkm$ (resp., $spor$, $utxv$).

## 5.4 Further research directions

The problems stated above can be extended to many POPs by inserting dash(es) in the SPOPs discussed. Also, a natural generalization of any avoidance problem is finding the distribution of a (S)POP under consideration. Moreover, joint distribution of (S)POPs and, as a special case, multi-avoidance of these patterns, is a possible direction for further research after choosing (S)POPs to consider. All these problems are interesting from enumerative point of view but also might bring interesting connections to other combinatorial objects, in which case, as always, explicit bijections would be desirable.

# 6 Acknowledgements

# References

[1] L. Aceto: Private communication (2006).

[2] E. Babson and E. Steingrímsson: Generalized permutation patterns and a classification of the Mahonian statistics, Séminaire Lotharingien de Combinatoire, B44b:18pp, 2000.

[3] P. Brändén, A. Claesson, and E. Steingrímsson: Catalan Continued Fractions and Increasing Subsequences in Permutations, *Discrete Math.* **258** (2002), 275–287

[4] A. Björner and M. L. Wachs: Permutation statistics and linear extensions of posets, *J. of Combin. Theory, Series A* **58** (1991), 85–114.

[5] A. Burstein and S. Kitaev: Partially ordered generalized patterns and their combinatorial interpretation, *The Third International Conference on Permutation Patterns*, University of Florida, Gainesville, Florida, March 7–11, 2005.

[6] A. Claesson: Generalised pattern avoidance, *European J. of Combin.* **22** (2001), 961–971.

[7] L. Comtet: *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht, 1974.

[8] S. Elizalde and M. Noy: Consecutive patterns in permutations, *Advances in Applied Math.* **30** (2003), 110–125.

[9] R. Entringer: Enumeration of permutations of $(1, \cdots, n)$ by number of maxima, *Duke Math. J.* **36** (1969), 575–579.

[10] G. Firro and T. Mansour: Restricted permutations and polygons, *The Third International Conference on Permutation Patterns*, University of Florida, Gainesville, Florida, March 7–11, (2005).

[11] P. Flajolet and R. Sedgewick: *An Introduction to the Analysis of AlgoriThm.s*, Addison-Wesley, 1996.

[12] I. Gessel and R. Stanley: Algebraic enumeration, *Handbook of combinatorics* (vol. **2**) (1996), 1021–1061.

[13] I. Golden and D. Jackson: *Combinatorial enumeration*, John Wiley,. New York, 1983.

[14] Q. Hou and T. Mansour: Horse paths, restricted 132-avoiding permutations, continued fractions, and Chebyshev polynomials, *Discrete Applied Mathematics* **154:8** (2006) 1183–1197.

[15] S. Kitaev: Multi-avoidance of generalised patterns, *Discrete Math.* **260** (2003), 89–100.

[16] S. Kitaev: Partially ordered generalized patterns, *Discrete Math.* **298** (2005), 212–229.

[17] S. Kitaev: Segmented partially ordered generalized patterns, *Theoretical Computer Science* **349** (2005) 3, 420–428.

[18] S. Kitaev and T. Mansour: Partially ordered generalized patterns and $k$-ary words, *Annals of Combin.* **7** (2003), 191–200.

[19] S. Kitaev, T. McAllister, K. Petersen: Enumerating segmented patterns in compositions and encoding with restricted permutations, preprint.

[20] D.E. Knuth: *The Art of Computer Programming*, Vol. 4 Fascicle 3, Generating All Combinations and Partitions (2005), vi+150pp. ISBN 0-201-85394-9

[21] T. Mansour: Restricted 1-3-2 permutations and generalized patterns, *Annals of Combin.* **6** (2002), 65–76.

[22] T. Mansour and A. Vainshtein: Restricted 132-avoiding permutations, *Adv. Appl. Math.* **126** (2001), 258–269.

[23] A. Mendes and J. B. Remmel: Permutations and words counted by consecutive patterns, *Advances in Appl. Math.* **37** (2006) 443–480.

[24] R. Rieper and M. Zeleke: Valleyless Sequences, *Congressus Numerantium* **145** (2000), 33–42.

[25] Sloane's On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences

[26] D. Warren and E. Seneta: Peaks and Eulerian numbers in a random sequence, *J. Appl. Probab.* **33(1)** (1996), 101–114.