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Theoretical Computer Science 349 (2005) 420–428

Theoretical Computer Science

www.elsevier.com/locate/tcs

Segmental partially ordered generalized patterns

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Received 15 October 2004; received in revised form 3 May 2005; accepted 12 September 2005

Communicated by D. Perrin

Abstract

We continue the study of partially ordered generalized patterns (POGPs) considered in [E. Babson, E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, Séminaire Lotharingien de Combinatoire, 2000, B44b:18pp] for permutations and in [A. Burstein, T. Mansour, Words restricted by patterns with at most 2 distinct letters, Electron. J. Combin. 9 (2) (2002) #R3] for words. We deal with segmental POGPs (SPOGPs). We state some general results and treat a number of patterns of length 4. We prove a result from [S. Kitaev, Multi-avoidance of generalized patterns, Discrete Math. 260 (2003) 89–100] in a much simpler way and also establish a connection between SPOGPs and walks on lattice points starting from the origin and remaining in the positive quadrant. We give a combinatorial interpretation of the powers of the (generalized) Fibonacci numbers. The entire distribution of the maximum number of non-overlapping occurrences of a generalized pattern with no dashes in permutations or words studied in [S. Kitaev, Partially ordered generalized patterns, Discrete Math. to appear, S. Kitaev, T. Mansour, Partially ordered generalized patterns and k-ary words, Ann. Combin. 7 (2003) 191–200], respectively, has its counterpart in case of SPOGPs.

Keywords: Pattern avoidance; Segmental patterns; Permutations; Words; Walks on lattice points; Distribution

1. Introduction

We write permutations as words $\pi = a_1a_2\cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$.

An occurrence of a *pattern* τ in a permutation π is "classically" defined as a subsequence in π (of the same length as τ) whose letters are in the same relative order as those in τ. For example, the permutation 31 425 has three occurrences of the pattern 123, namely the subsequences 345, 145, and 125. Considering occurrences of patterns in permutations has its roots in the works by Rotem, Rogers, and Knuth in the 1970s and early 1980s.

In [1] Babson and Steingrímsson introduced *generalized permutation patterns* that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in a permutation π, then the

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[1] This paper was written while the author was visiting the University of Kentucky.

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2005.09.036

letters in π that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516\,423$ has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563.

The motivation for introducing these patterns in [1] was the study of Mahonian statistics. A number of interesting results on generalized patterns were obtained in [5]. Relations to several well-studied combinatorial structures, such as set partitions, Dyck paths, Motzkin paths and involutions, were shown there. However, the *segmental patterns* are of particular interest in this paper, which are the patterns with no dashes, that is, patterns correspond to contiguous subwords anywhere in a permutation (in some literature, we meet *subword patterns*, *contiguous patterns*, and *patterns with no gaps* meaning the same concept). Such patterns were considered, e.g., in [6,8].

In [9] introduced a further generalization of generalized patterns (GPs), namely *partially ordered generalized patterns* (*POGPs*). A POGP is a GP some of whose letters are incomparable. For instance, if we write $p = 1\text{-}1'2'$ then we mean that in an occurrence of p in a permutation π the letter corresponding to the 1 in p can be either larger or smaller than the letters corresponding to $1'2'$. Thus, the permutation 31 254 has three occurrences of p, namely 3-12, 3-25, and 1-25. A motivation for introducing POGPs is that they allow us to find the *exponential generating function e.g.f.* for the *entire distribution* of the maximum number of non-overlapping occurrences of a pattern p with no dashes, if we only know the e.g.f. $A(x)$ for the number of permutations that avoid p (we do know the e.g.f. for a lot of GPs with no dashes due to [6]). This distribution, according to [9, Theorem 32], is given by

$$\frac{A(x)}{1 - y - y(x-1)A(x)}.$$

In [11], the concept of POGPs in permutations is extended to that in words. Let $[k]^n$ denote the set of all the words of length n over the (totally ordered) alphabet $[k] = \{1, 2, \ldots, k\}$. For example, if we write $\tau = 1\text{-}1'2'$, then we mean that in an occurrence of τ in a word $\sigma \in [k]^n$ the letter corresponding to the 1 in τ can be either larger, smaller, or equal to the letters corresponding to $1'$ and $2'$, whereas the letter corresponding to $1'$ must be less than the letter corresponding to $2'$. Thus, the word $113\,425 \in [5]^6$ contains seven occurrences of τ, namely 113, 134 twice, 125 twice, 325, and 425. Note, that a crucial difference with the case of permutations is that our patterns in words might have repeated letters. For instance, the pattern $\tau = 111'2'$ makes sense in the case of words, but not in the case of permutations.

The main result in [11] is the *generating function* (*g.f.*) for the entire distribution of the maximum number of non-overlapping occurrences of a pattern τ with no dashes (and possibly with repeated letters) in k-ary words, provided we know the g.f. $A_\tau(x; k)$ for the number of k-ary words that avoid τ (we do know the g.f. for many of such patterns due to [2–4]). This distribution, according to [11], is given by

$$\frac{A_\tau(x;k)}{1 - y - y(kx-1)A_\tau(x;k)}.$$

In the avoidance problems, any POGP may be viewed as a convenient short notation for a certain set of GPs. Indeed, to avoid a POGP is the same as to avoid simultaneously a number of certain GPs. For example, to avoid the POGP $21'1$ is the same as to avoid simultaneously the GPs 231, 312, and 321. From this point of view, study of POGPs is a continuation of study of multi-avoidance of usual GPs. In particular, results for the *segmental POGPS* (*SPOGPs*) considered in this paper are in the same direction as the results from [8] and some of the results from [6,9]. A POGP is segmental if its occurrence in a permutation forms a contiguous subword of the permutation of the corresponding length.

In Section 2 we state two results (see Proposition 1 and Corollary 2) that are helpful when dealing with SPOGPs, and they are used in a few places throughout the paper.

One SPOGP of length four, namely $122'1'$, was considered in [9, Section 6] (see Proposition 3). In Section 2.1 we study the number of permutations avoiding some other SPOGPs of length four. One of these POGPs, namely $12'21'$, give us a connection to walks on lattice points starting from the origin and remaining in the non-negative quadrant (see Proposition 9 in Section 2.2).

In [8], when finding the number of n-permutations avoiding simultaneously the GPs 123, 132, and 213, which is $A_n = \binom{n}{\lfloor n/2 \rfloor}$, rather complicated considerations were used. In Section 3, we introduce *separated* SPOGPs (SSPOGPs), and give a “two line” proof of the result for A_n.

Although this paper is primarily concerned about occurrences of SPOGPs in permutations, in Section 4 we extend a result on SSPOGPs for permutations to that for words. In particular, we give a combinatorial interpretation of the powers of the (generalized) Fibonacci numbers.

Finally, in Section 5 we discuss the distribution of the maximum number of non-overlapping occurrences of SPOGPs in permutations and words. It turns out that the bivariate (exponential) generating functions for this distribution is given by the formulas stated above, and thus, once we solved the avoidance problem for a SPOGP, we are done with the maximum number of non-overlapping occurrences problem for this SPOGP. To explain this result, we introduce *multi-patterns* for SPOGPs which are a generalization of the multi-patterns considered in [9] for permutations and in [11] for words.

2. Segmental POGPs (SPOGPs)

Recall that a POGP is segmental, if its occurrence in a permutation form a contiguous subword of the permutation.

Let $V(p)$ denote the alphabet of variables of the pattern p.

Let p be a SPOGP. A permutation π *quasi-avoids* p if π has exactly one occurrence of p and this occurrence consists of the $|p|$ rightmost letters of π. The concept of quasi-avoidance is helpful under certain enumeration problems (e.g., see the proof of Corollary 2).

Proposition 1. *Let p be a SPOGP and $P(x)$ (resp. $P^*(x)$) be the e.g.f. for the number of permutations that avoid (resp. quasi-avoid) p. Then*

$$P^*(x) = (x-1)P(x) + 1.$$

Proof. One can copy the proof of [9, Proposition 4]. However, an alternative proof can be given. Let P_i (resp. P_i^*) be the number of i-permutations that avoid (resp. quasi-avoid) p. Also, suppose $|p| = k$, that is p consists of k letters. Then we first count the number of n-permutations containing p and subtract this number from $n!$ to get the desired.

Any permutation π containing p can be uniquely factored as $\pi = \pi_1\pi_2$, where π_1 quasi-avoids p and $|\pi_1| = i$. Clearly, $k \leqslant i \leqslant n$, and $P_j^* = 0$, for $1 \leqslant j \leqslant k-1$. Thus,

$$P_n = n! - \sum_{i=1}^{n} \binom{n}{i} P_i^*(n-i)!,$$

that is $P(x) = 1/(1-x) - P^*(x)/(1-x)$, which completes our proof. □

Corollary 2. *Let p be a SPOGP, the letter $a \notin V(p)$, and a is incomparable to any $b \in V(p)$. If $P(x)$ (resp. $P_a(x)$) is the e.g.f. for the number of permutations that avoid p (resp. the SPOGP pa) then $P_a(x) = xP(x) + 1$.*

Proof. The permutations avoiding pa are those avoiding p or quasi-avoiding p. Thus, according to Proposition 1,

$$P_a(x) = P(x) + (x-1)P(x) + 1 = xP(x) + 1. \qquad \square$$

2.1. Segmental POGPs of length four

The standard reverse and complement operations on permutations divide the patterns into equivalence classes. We consider the following representatives of several (but not all) equivalence classes of SPOGPs of length four : $12'21'$, $11'22'$, $122'1'$, $121'2'$, $11'2'2$, $12'1'2$, $1231'$, $1321'$, $2131'$, $121'3$, $131'2$, $231'1$, $1'1''12$, $1'11''2$, $1'121''$, and $11'1''2$. Note, that each of the three last patterns corresponds to simultaneous avoidance of two SPOGPs. For example, to avoid $1'1''12$ is the same as to avoid $1'2'12$ and $2'1'12$ simultaneously. Currently, we do not know solutions to the avoidance problem for the equivalence classes having the patterns $11'22'$, $121'2'$, $11'2'2$, $12'1'2$, $121'3$, $131'2$, and $231'1$, as well as the other equivalence classes of SPOGPs of length four that are not mentioned above. We record a few initial values for the number of n-permutations avoiding the stated unsolved patterns, $n \geqslant 1$, in Table 1, and we leave consideration of these patterns as a challenging problem.

Table 1
The initial values for the number of n-permutations avoiding 4-SPOGPs in the unsolved cases, $n \geqslant 1$

$11'22'$	1, 2, 6, 18, 70, 300, 1435, 7910, 47 376, ...
$121'2'$	1, 2, 6, 18, 61, 281, 1541, 8920, 57 924, ...
$11'2'2$	1, 2, 6, 18, 71, 322, 1665, 9789, 64 327, ...
$12'1'2$	1, 2, 6, 18, 61, 272, 1410, 8048, 51 550, ...
$121'3$	1, 2, 6, 20, 83, 411, 2290, 14 588, 104 448, ...
$131'2$	1, 2, 6, 20, 81, 390, 2161, 13 678, 96 983, ...
$231'1$	1, 2, 6, 20, 83, 402, 2245, 14 192, 100 650, ...

In all the propositions in this section, we assume B_n (resp. $B(x)$) denotes the number (resp. the e.g.f. for the number) of permutations that avoid a pattern under consideration.

The following result was obtained in [9].

Proposition 3 (*Kitaev [9, Theorem 30]*). *Let* $p = 122'1'$. *Then*

$$B(x) = \frac{1}{2} + \frac{1}{4}\tan x(1 + \mathrm{e}^{2x} + 2\mathrm{e}^{x}\sin x) + \frac{1}{2}\mathrm{e}^{x}\cos x.$$

We now consider some other patterns.

Proposition 4. *For the pattern* $11'21''$ *and* $n \geqslant 1$, *we have* $B_n = n \cdot \binom{n-1}{\lfloor (n-1)/2 \rfloor}$.

Proof. Clearly, to avoid $11'21''$ is the same as either to avoid the pattern $11'2$ or to quasi-avoid $11'2$. If A_n denotes the number of permutations that avoid $11'2$ then according to the way the proof of [9, Proposition 4] goes, and to the considerations in the beginning of Section 3, where we find A_n, we have

$$B_n = A_n + A^* = A_n + nA_{n-1} - A_n = n \cdot \binom{n-1}{\lfloor (n-1)/2 \rfloor}. \qquad \square$$

Proposition 5. *For the pattern* $1'1''12$ *and* $1'121''$, *we have* $B_0 = B_1 = 1$, *and, for* $n \geqslant 2$, $B_n = n(n-1)$.

Proof. If $\pi = \pi_1\pi_2\cdots\pi_n$ avoids $1'1''12$ then there are no restrictions for $\pi_1\pi_2$ and $\pi_3\pi_4\cdots\pi_n$ must be in decreasing order.

If $\pi = \pi_1\pi_2\cdots\pi_n$ avoids $1'121''$ then there are no restrictions for π_1 and π_n, and $\pi_2\pi_3\cdots\pi_{n-1}$ must be in decreasing order. $\square$

Proposition 6. *For the pattern* $11'1''2$, *we have that*

$$B_n = \frac{n!}{\lfloor n/3 \rfloor! \lfloor (n+1)/3 \rfloor! \lfloor (n+2)/3 \rfloor!}.$$

Proof. As a corollary to Theorem 11, with $\ell = 2$ and $A_n = 1$, we get that

$$B_n = \prod_{i=0}^{m_1-1} \binom{n - k_1 \cdot i}{k_1} \prod_{j=0}^{2-m_1} \binom{n - k_1 \cdot m_1 - k_2 \cdot j}{k_2},$$

where $k_1 = \lceil n/3 \rceil$, $k_2 = \lfloor n/3 \rfloor$, and $m_1 = n - 3 \cdot \lfloor n/3 \rfloor$. This expression for B_n can be seen to be equal to that we need to prove, by, e.g., checking the cases: $n = 3k$, $n = 3k+1$, and $n = 3k+2$. The initial values for B_n are $1, 2, 6, 12, 30, 90, 210, 560, 1680, 4200, \ldots$.

Another proof of the proposition is observing that the subsequences $\pi_1\pi_4\pi_7\cdots$, $\pi_2\pi_5\pi_8\cdots$, and $\pi_3\pi_6\pi_9\cdots$ must be in decreasing order. Thus we take all $n!$ permutations and divide them by the product of the numbers of permutations corresponding to the subsequences considered above. $\square$

Proposition 7. *For the pattern* $1231'$, *we have that*

$$B(x) = xe^{x/2}\left(\cos\frac{\sqrt{3}x}{2} - \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}x}{2}\right)^{-1} + 1,$$

and for the patterns $1321'$ *and* $2131'$, *we have that*

$$B(x) = x\left(1 - \int_0^x e^{-t^2/2}\,dt\right)^{-1} + 1.$$

Proof. According to Proposition 2, our aim is to find the e.g.f.s for the number of permutations avoiding the segmental patterns 123, 132, and 213, which are given by Kitaev [9, Theorems 11, 12] and the discussions after those theorems in [9, Section 3]. $\square$

The following proposition follows from Proposition 9 and formula (1.14) of [7]. However we provide an alternative, direct proof of the proposition.

Proposition 8. *For the pattern* $12'21'$, *we have that*

$$B_n = \binom{n-1}{\lfloor (n-1)/2\rfloor}\binom{n}{\lfloor n/2\rfloor}.$$

Proof. Suppose an n-permutation π avoids $12'21'$.

If the letter 1 is in position t, that is $\pi(t) = 1$, then $\pi(t-2)\pi(t)$ (resp. $\pi(t)\pi(t+2)$) forms the pattern 21 (resp. 12). Since π avoids $12'21'$, $\pi(t-3)\pi(t-1)$ (resp. $\pi(t+1)\pi(t+3)$) must form the pattern 21 (resp. 12), which leads to $\pi(t-4)\pi(t-2)$ (resp. $\pi(t+2)\pi(t+4)$) must form the pattern 21 (resp. 12), and so on. This shows that the letters to the left (resp. to the right) of 1 having the same position parity as 1 does must decrease (resp. increase). The same holds for the letters to the left and to the right of 1 having the different position parity. There are no additional restrictions.

We distinguish two cases: n is even and n is odd. In each of these cases we consider two subcases: 1 is in an even position and it is in an odd position.

(1) *n is even, $\pi^{-1}(1)$ is odd*: we choose the letters in the even positions in $\binom{n-1}{n/2}$ ways. Suppose now that 1 is in the $(i+1)$st odd position, where $0 \leqslant i \leqslant n/2-1$. We choose the letters in the odd positions to the left of 1 in $\binom{n/2-1}{i}$ ways, and we choose the letters in the even positions to the left of 1 in $\binom{n/2}{i}$ ways. Then we order all the letters uniquely according to the considerations above.

Similarly, we get the other cases.

(2) *n is even, $\pi^{-1}(1)$ is even*: we have $\binom{n-1}{n/2}\sum_{i=0}^{n/2-1}\binom{n/2-1}{i}\binom{n/2}{i+1}$ permutations;

(3) *n is odd, $\pi^{-1}(1)$ is odd*: we have $\binom{n-1}{(n-1)/2}\sum_{i=0}^{(n-1)/2}\binom{(n-1)/2}{i}\binom{(n-1)/2}{i}$ permutations;

(4) *n is odd, $\pi^{-1}(1)$ is even*: we have $\binom{n-1}{(n-1)/2-1}\sum_{i=0}^{(n-1)/2-1}\binom{(n-1)/2-1}{i}\binom{(n+1)/2}{i+1}$ permutations.

Thus, B_n is given by

$$\begin{cases} \binom{n-1}{n/2}\left(\sum\limits_{i=0}^{n/2-1}\binom{n/2-1}{i}\binom{n/2}{i} + \sum\limits_{i=0}^{n/2-1}\binom{n/2-1}{i}\binom{n/2}{i+1}\right), & n \text{ is even,}\\ \binom{n-1}{(n-1)/2}\sum\limits_{i=0}^{(n-1)/2}\binom{(n-1)/2}{i}^2 + \binom{n-1}{(n+1)/2}\sum\limits_{i=0}^{(n-3)/2}\binom{(n-3)/2}{i}\binom{(n+1)/2}{i+1}, & n \text{ is odd.}\end{cases}$$

To prove the statement, we need to prove that

$$\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{\lfloor\frac{n}{2}\rfloor} = \binom{n-1}{\lfloor\frac{n}{2}\rfloor}\sum_{i=0}^{\lceil\frac{n}{2}\rceil-1}\binom{\lceil\frac{n}{2}\rceil-1}{i}\binom{\lfloor\frac{n}{2}\rfloor}{i} + \binom{n-1}{\lceil\frac{n}{2}\rceil}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor-1}\binom{\lfloor\frac{n}{2}\rfloor-1}{i}\binom{\lceil\frac{n}{2}\rceil}{i+1},$$

where the right-hand side is just rewritten expression for B_n.

If $n = 2k$ for some $k \geqslant 1$, then the equality to prove reduces to

$$\binom{2k}{k} = \sum_{i=0}^{k-1}\binom{k-1}{i}\binom{k+1}{i+1} = \sum_{i=0}^{k-1}\binom{k-1}{i}\binom{k+1}{k-i},$$

which is true using, e.g., a combinatorial proof with choosing a committee consisting of k people out of $k-1$ men and $k+1$ women.

If $n = 2k+1$ for some $k \geqslant 0$, then the equality to prove reduces to

$$\binom{2k}{k}\binom{2k+1}{k} = \binom{2k}{k}\sum_{i=0}^{k}\binom{k}{i}^2 + \binom{2k}{k+1}\sum_{i=0}^{k-1}\binom{k-1}{i}\binom{k+1}{i+1},$$

which holds since $\binom{2k}{k} = \sum_{i=0}^{k-1}\binom{k-1}{i}\binom{k+1}{i+1} = \sum_{i=0}^{k}\binom{k}{i}\binom{k}{k-i}$, and $\binom{2k+1}{k} - \binom{2k}{k+1} = \binom{2k}{k}$. □

2.2. Segmental POGPs and walks

We refer to [12] for a survey on counting walks in sectors of the plane. Many interesting results on walks can be found in [7].

Proposition 9. *The number of* $(n+1)$*-permutations avoiding the SPOGP* $12'21'$ *is equal to the number of different walks of n steps between lattice points, each in a direction N, S, E or W, starting from the origin and remaining in the non-negative quadrant.*

Proof. We give a combinatorial proof of the statement by providing a bijection between the $(n+1)$-permutations avoiding $12'21'$ and the walks of n steps in question. [2] Clearly, any such walk can be coded by a word $w = w_1w_2\cdots w_n$ over the alphabet $\{a, \bar{a}, b, \bar{b}\}$ with the property that for any i, $1 \leqslant i \leqslant n$, the number of a's (resp. b's) in $w_1w_2\cdots w_i$ is not less than the number of $\bar{a}$'s (resp. $\bar{b}$'s) there. We may assume that a is the direction N, $\bar{a}$–S, b–E, and $\bar{b}$–W.

We use the *Catalan factorization* (see [12, Lemma 9.1.1]) of a word on two letters together with a simple bijection. Consider a word w over the alphabet $\{a, \bar{a}, b, \bar{b}\}$ as a two-line array where the top line is an increasing sequence of positions $1, 2, \ldots, |w|$ and the bottom line is the word w in one-line notation. Let w_a be the subarray of w on letters $\{a, \bar{a}\}$. Consider the Catalan factorization of $w_a = D_a^{(1)}aD_a^{(2)}aD_a^{(3)}\ldots$, where the bottom line of each $D_a^{(i)}$ is a Dyck word on $\{a, \bar{a}\}$. Permute the columns of w_a to obtain a two-line array $w'_a = T_a^{(1)}aT_a^{(2)}aT_a^{(3)}\ldots$ such that each $D_a^{(i)}$ maps onto $T_a^{(i)}$ of the same length with the bottom line $(\bar{a}a)^*$, and positions of a's and $\bar{a}$'s in w_a increase left-to-right on the top line of w'_a. Now let π_a be the top line of w'_a. Note that the positions of descent tops of π_a are exactly the positions of $\bar{a}$'s in w. Similarly, let w_b be the subarray of w on letters $\{b, \bar{b}\}$ and obtain the permutation π_b as above. Define $\pi = [r(\pi_a+1)]1[\pi_b+1]$, where r is the reversal map, and π_a+1 (resp. π_b+1) denotes a permutation obtained by adding 1 to each entry of π_a (resp. π_b). This is easy to see using the structure of "good" permutations described in the proof of Proposition 8 that π is an $(n+1)$-permutation avoiding $12'21'$. Also, the described map is injective and its inverse is easy to construct. □

3. Separated segmental POGPs (SSPOGPs)

Definition 10. Suppose $p = a_1a_2\cdots a_k$ is a permutation and, for fixed non-negative integers $\ell_1, \ell_2, \ldots, \ell_{k-1}$, the letters $b^{(i,j)}$, $1 \leqslant i \leqslant k-1$, $1 \leqslant j = j(i) \leqslant \ell_i$, are incomparable neither with each other nor with a_is, $1 \leqslant i \leqslant k$. We call the SPOGP

$$a_1b^{(1,1)}b^{(1,2)}\ldots b^{(1,\ell_1)}a_2b^{(2,1)}b^{(2,2)}\ldots b^{(2,\ell_2)}a_3\cdots a_{k-1}b^{(k-1,1)}b^{(k-1,2)}\ldots b^{(k-1,\ell_{k-1})}a_k$$

[2] The proof we provide here was suggested by the referee. For another, longer though, proof using the "jumping procedure" see [10].

separated segmental POGP (*SSPOGP*). For the SSPOGP above we use the notation

$$\tau_k(\ell_1, \ell_2, \dots, \ell_{k-1}) = a_1|_{\ell_1}a_2|_{\ell_2}a_3 \cdots a_{k-1}|_{\ell_{k-1}}a_k.$$

We use "|" instead of "$|_1$".

The idea of considering patterns in permutations having the properties like the pattern $\tau_k(\ell_1, \ell_2, \dots, \ell_{k-1})$ has, that is when in any occurrence of a pattern in permutations, the distance between any two letters from this occurrence is a fixed positive number, does not seem to be new. However, it might be useful to realize that such patterns form a subclass of POGPs.

We have already met SSPOGPs in Section 2.1, e.g., $p = 121'3$ (here $\ell_1 = 0$ and $\ell_2 = 1$). However, the easiest example of a SSPOGP is the pattern $11'2 = 1|2$. There are $\binom{n}{\lfloor n/2 \rfloor}$ permutations avoiding this pattern. Indeed, we choose the letters of our permutation in odd positions in $\binom{n}{\lfloor n/2 \rfloor}$ ways, and we must arrange them in decreasing order. We then must arrange the letters in even positions in decreasing order too. We used the property that the letters in odd and even positions do not affect each others.

A rather simple argument above, when considering the pattern 1|2, considerably simplifies the proof of [8, Theorem 1] after observing that to avoid 1|2 is the same as to avoid simultaneously the SPOGPs 123, 132, and 213. The simple idea of separating a permutation into parts that do not affect each others gives the number of permutations that avoid a SSPOGP having $\ell_i = \ell_j$ for all i, j. We record this result into the following theorem which is easy to prove using the observation on separation.

Theorem 11. *Let A_n denote the number of n-permutations avoiding the segmental pattern $a_1a_2 \cdots a_k$; $k_1 = \lceil n/(\ell + 1) \rceil$, $k_2 = \lfloor n/(\ell + 1) \rfloor$, $m_1 = n - (\ell + 1)\lfloor n/(\ell + 1) \rfloor$, and $m_2 = (\ell + 1)(\lfloor n/(\ell + 1) \rfloor + 1) - n$. Then, for $\ell \geqslant 0$, the number B_n of n-permutations avoiding the SSPOGP $\tau_{k,\ell} = a_1|_\ell a_2|_\ell a_3 \cdots a_{k-1}|_\ell a_k$ is given by*

$$A_{k_1}^{m_1} A_{k_2}^{m_2} \prod_{i=0}^{m_1-1} \binom{n - k_1 \cdot i}{k_1} \prod_{j=0}^{m_2-1} \binom{n - k_1 \cdot m_1 - k_2 \cdot j}{k_2}.$$

Proof. When using the idea of separation, clearly there are m_1 subsequences of length k_1 and m_2 subsequences of length k_2. The rest is easy to see. □

Remark 12. If n is devisible by $\ell + 1$, the result of Theorem 11 can be simplified as

$$B_n = A_{n/(\ell+1)}^{\ell+1} \prod_{i=0}^{\ell} \binom{n(\ell - i + 1)/(\ell + 1)}{n/(\ell + 1)}.$$

In particular, if $\ell = 0$ then $B_n = A_n$.

Remark 13. We can use the result in Theorem 11 to get an explicit number of permutations avoiding $\tau_{k,\ell}$ in a lot of particular cases due to the results in [6,9] (we simply use explicit values of A_n whenever we know them).

4. Separated segmental POGPs and words

Similar to Definition 10 one can define a separated segmental POGP when considering words rather than permutations. We may also allow $a_1a_2 \cdots a_k$ from Definition 10 to be a word.

An analogue of Theorem 11 is the following theorem:

Theorem 14. *Let $A(n; p)$ denote the number of all words from $[p]^n$ avoiding the segmental pattern $a_1a_2 \cdots a_k$ over $[m]$; $k_1 = \lceil n/(\ell + 1) \rceil$, $k_2 = \lfloor n/(\ell + 1) \rfloor$, $m_1 = n - (\ell + 1)\lfloor n/(\ell + 1) \rfloor$, and $m_2 = (\ell + 1)(\lfloor n/(\ell + 1) \rfloor + 1) - n$. Then, for $\ell \geqslant 0$, the number $B(n; p)$ of words from $[p]^n$ that avoid the SSPOGP $\tau_{k,\ell,m} = a_1|_\ell a_2|_\ell a_3 \cdots a_{k-1}|_\ell a_k$ is given by $A^{m_1}(k_1; p)A^{m_2}(k_2; p)$.*

Remark 15. We can use the result in Theorem 14 to get explicit numbers of words over a p-letter alphabet avoiding $\tau_{k,\ell,m}$ in many particular cases due to the results in [2–4] (we can use explicit values of $A(n; p)$ whenever we know them).

This is well known and it is not difficult to see that the number of different binary strings of length n that avoid the segmental pattern 11 is given by F_{n+2}, where F_n is the nth Fibonacci number defined by $F_0 = F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geqslant 0$. This fact alone with Theorem 14 give us a combinatorial interpretation of the powers of the Fibonacci numbers. Indeed, for $n \geqslant 2$ and $\ell \geqslant 0$, the number of binary words of length $(\ell + 1) \cdot (n - 2)$ avoiding the SSPOGP $1|_\ell 1$ is given by $F_n^{\ell+1}$. Likewise, the number of binary words of certain lengths avoiding a SSPOGP pattern $1|_\ell 1 \cdots 1|_\ell 1$ is given by a certain power of a *generalized Fibonacci number*.

5. The distribution of non-overlapping SPOGPs

We define a *multi-pattern* to be a pattern $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_t$, where $\{\sigma_1, \sigma_2, \ldots, \sigma_t\}$ is a set of SPOGPs and each letter of σ_i is incomparable with any letter of σ_j whenever $i \neq j$. Our multi-patterns are a generalization of the multi-patterns considered in [9] for permutations and in [11] for words.

It turns out that we can copy all the arguments from [9,11], when considering the multi-patterns there, to get results for our multi-patterns. Indeed, the fact that in a pattern σ_i, for some i, some of the letters might be incomparable does not affect any of the considerations. In particular, [9, Theorem 28] (resp. [11, Theorem 4.1]) gives a way to obtain the e.g.f. (resp. g.f.) for the number of permutations (resp. words) avoiding a multi-pattern $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_t$, provided we know the e.g.f. (resp. g.f.) for the number of permutations (resp. words) that avoid the SPOGP σ_i for each i, $1 \leqslant i \leqslant t$.

Theorem 16 gives an interesting application of the multi-patterns in finding a certain statistic, namely the *maximum number of non-overlapping occurrences of a SPOGP* in permutations and words. For instance, the maximum number of non-overlapping occurrences of the SPOGP $11'2$ in the permutation 621 394 785 is 2, which is given by the occurrences 213 and 478, or the occurrences 139 and 478.

Theorem 16 can be proven in the same way its counterparts [9, Theorem 32; 11, Theorem 5.1] were proven.

Theorem 16. *Let p be a SPOGP. Let $B(x)$ (resp. $B(x; k)$) be the e.g.f. (resp. g.f.) for the number of permutations (resp. words over $[k]$) that avoid p. Let $D(x, y) = \sum_\pi y^{N(\pi)} \frac{x^{|\pi|}}{|\pi|!}$ and $D(x, y; k) = \sum_{n\geqslant 0} \sum_{w\in[k]^n} y^{N(w)} x^n$ where $N(s)$ is the maximum number of non-overlapping occurrences of p in s. Then*

$$D(x, y) = \frac{B(x)}{1 - y(1 + (x - 1)B(x))}$$

and

$$D(x, y; k) = \frac{B(x; k)}{1 - y(1 + (kx - 1)B(x; k))}.$$

The following examples are corollaries to Theorem 16.

Example 1. If we consider the SPOGP $11'$ then clearly $B(x) = 1 + x$ and $B(x; k) = 1 + kx$. Hence,

$$D(x, y) = \frac{1 + x}{1 - yx^2} = \sum_{i\geqslant 0} (x^{2i} + x^{2i+1})y^i,$$

and

$$D(x, y; k) = \frac{1 + kx}{1 - y(kx)^2} = \sum_{i\geqslant 0} ((kx)^{2i} + (kx)^{2i+1})y^i,$$

which is easy to see to be true.

Example 2. If we consider permutations and the SPOGP $122'1'$ then $B(x)$ is given by Proposition 3, and the distribution of the maximum number of non-overlapping occurrences of $122'1'$ is given by the formula

$$D(x,y)=\frac{\frac{1}{2}+\frac{1}{4}\tan x(1+\mathrm{e}^{2x}+2\mathrm{e}^{x}\sin x)+\frac{1}{2}\mathrm{e}^{x}\cos x}{1-y(1+(x-1)(\frac{1}{2}+\frac{1}{4}\tan x(1+\mathrm{e}^{2x}+2\mathrm{e}^{x}\sin x)+\frac{1}{2}\mathrm{e}^{x}\cos x))}.$$

If we are interested in, say, just one non-overlapping occurrence of $122'1'$, we consider the coefficient of y in the expansion of $D(x, y)$:

$$\frac{1}{4}x^4+\frac{9}{20}x^5+\frac{13}{20}x^6+\frac{23}{30}x^7+\frac{143}{180}x^8+\frac{301}{405}x^9+\frac{2591}{4050}x^{10}+\cdots.$$

That is, the initial values for the number of permutations having one non-overlapping occurrence of $122'1'$ are 0, 0, 0, 0, 6, 54, 468, 3864, 32 032, 269 696, 2 321 536, ...

Acknowledgments

The author is grateful to Richard Ehrenborg, Carl Lee, Margaret Readdy, and Peter Perry for their support during the author's stay in the Mathematics Department at the University of Kentucky. He is also grateful to the referee for useful remarks and for providing a simpler proof of Proposition 9.

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