# On the number of self-dual rooted maps *

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**Abstract**

We compute the number of self-dual rooted maps with $n$ edges. We also compute the number of 2-connected and 3-connected self-dual rooted maps, and show that the later are counted by the Fine numbers.

## 1 Introduction

Let $G$ be a graph embedded in the plane and let $G^*$ be its dual graph. We say that $G$ is *self-dual* if $G$ and $G^*$ are isomorphic as embedded planar graphs, that is, if there exists an orientation-preserving homeomorphism of the sphere taking $G$ into $G^*$. Notice that we are not asking merely that $G$ and $G^*$ are isomorphic as graphs, they must be the same embedded plane graph. There are a number of references in the literature on self-dual planar graphs, mainly on how to produce them [1, 6, 7]. We remark that in these references, as opposed to our definition, $G$ and $G^*$ ar allowed to differ by a reflection.

In this paper we are interested in counting self-dual plane graphs. To this end we consider rooted planar maps instead of plane graphs. A *map* is a connected planar graph embedded in the sphere. A map is *rooted* if an edge $e = vw$ is distinguished together with a vertex $v$ on $e$ and a face $f$ incident with $e$. We follow the convention that $f$ is the face to the right of $e$ going from $v$ to $w$. We call $e, v$ and $f$, respectively, the root edge, vertex and face. The root edge is marked by an arrow on $e$ going from the tail $v$ to the head $w$. Maps are allowed to have loops and multiple edges. Two rooted maps are isomorphic if there is a homeomorphism of the sphere taking one map into the other, preserving incidences between vertices, edges and faces, and preserving the root vertex, edge and face. We recall that maps can be defined in a purely combinatorial way by means of the cyclic ordering of the edges around each vertex [9].

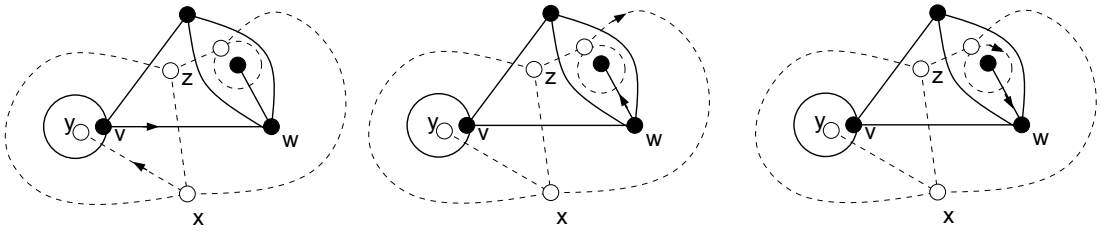


Figure 1: Rooting the dual map.

If $M$ is a rooted map, we define the dual map $M^*$ as follows. As a plane graph, $M^*$ is the dual plane graph of $M$. If $e = vw$ is the root edge of $M$, then the root edge of $M^*$ is $xy$, where $x$ corresponds to the root face of $M$, and $xy$ is defined as follows. Let $e^* = xz$ be the edge of

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$M^*$ crossed by $e$. Then take as $xy$ the edge following $xz$ in counter-clockwise order. Notice that in this way, the root vertex and face of $M^*$ correspond, respectively, to the root face and vertex of $M$. This rule has to be followed too in the special case when $e^*$ is a loop (equivalently, when $e$ is an isthmus). See Figure 1 for an illustration, where vertices of $M^*$ are white and edges are dashed. It is easy to check that with this definition duality is an involution on rooted maps, that is, $M^{**}$ and $M$ are isomorphic as rooted maps. A rooted map $M$ is *self-dual* if $M$ and $M^*$ are isomorphic.

All maps in this paper are rooted, so that we just speak of maps. We consider arbitrary (connected), 2-connected and 3-connected maps. A map is 2-connected if it has no loops and no cut vertices or it has only one edge (either a loop or an isthmus). It is 3-connected if in addition has no multiple edges and no separators of size less than three. We remark that 2-connected maps are also called non-separable, and 3-connected maps are called polyhedral. Since $k$-connectivity, for $k = 1, 2, 3$, is preserved under duality, it makes sense to consider self-dual maps in these classes. If a map $M$ is self-dual, then it has the same number of vertices as faces and, by Euler's formula, the number of edges is even. If $M$ has $2n$ edges, then it has $n+1$ vertices and $n+1$ faces.

Let us recall the following classical formulas [8] for the numbers $M_n$ and $B_n$, respectively, of maps and 2-connected maps with $n$ edges:

$$\begin{aligned} M_n &= \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}, \qquad n \geq 0, \\ B_n &= \frac{2}{n(3n-2)} \binom{3n-2}{2n-1}, \qquad n \geq 1. \end{aligned} \tag{1}$$

The generating function $M(z) = \sum M_n z^n$ satisfies the equation

$$M(z) = 1 - 16z + 18zM(z) - 27z^2 M(z)^2. \tag{2}$$

For the number $T_n$ of 3-connected maps we do not have a closed formula, but one can compute the generating function $T(z) = \sum T_n z^n$, whose first terms are

$$T(z) = z^6 + 4z^8 + 6z^9 + 24z^{10} + 214z^{11} + \cdots$$

We also need to recall the Fine numbers $F_n$, which are defined through their generating function

$$F(z) = \sum_{n \geq 0} F_n z^n = \frac{1 + 2z - \sqrt{1-4z}}{2z(z+2)} = 1 + z^2 + 2z^3 + 6z^4 + 18z^5 + 57z^6 + \cdots \tag{3}$$

They are related to the Catalan numbers $C_n$ through the recurrence $C_n = 2F_n + F_{n-1}$ (see [3]).

Our main results are summarised in the following theorem.

**Theorem 1.**

1. *For $n \geq 3$, the number of 3-connected self-dual maps with $2n$ edges equals the Fine number $F_{n-1}$.*
2. *For $n \geq 1$, the number of 2-connected self-dual maps with $2n$ edges is equal to*
$$\frac{1}{n} \binom{3n-2}{n-1}.$$
3. *For $n \geq 1$, the number of self-dual maps with $2n$ edges is equal to*
$$\frac{3^n}{n+1} \binom{2n}{n}.$$

The next table shows the number of self-dual maps with at most 12 edges.

| Number of edges | 3-connected | 2-connected | Arbitrary |
|---|---|---|---|
| 2 | | 1 | 3 |
| 4 | | 2 | 18 |
| 6 | 1 | 7 | 135 |
| 8 | 2 | 30 | 1134 |
| 10 | 6 | 143 | 10206 |
| 12 | 18 | 728 | 96228 |

Figure 2 shows a catalogue of self-dual maps with small number of edges. When a map has two arrows, it means it can be rooted in two non-isomorphic ways as a self-dual map. Observe that a map and its reflection though a line are considered different if they are not isomorphic in the oriented sphere.

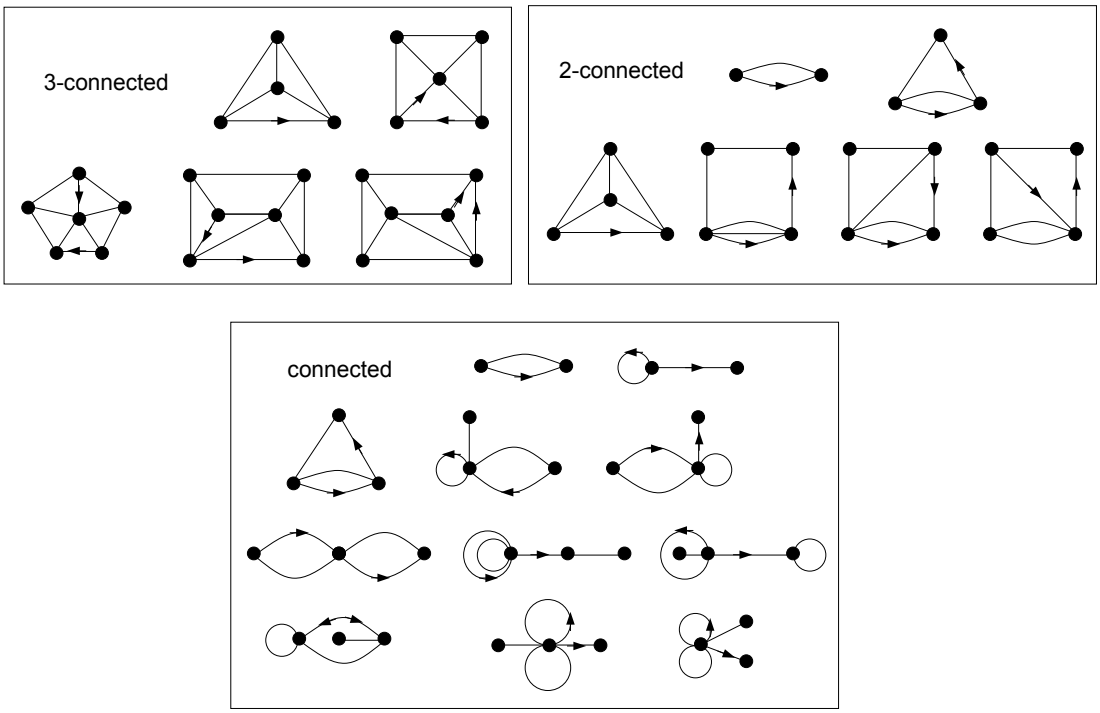


Figure 2: A catalogue of small self-dual maps.

From Theorem 1 it is easy to derive asymptotic estimates for the number of self-dual maps.

**Corollary 2.** *The number of self-dual maps with* $2n$ *edges is asymptotically*

$$\frac{1}{9\sqrt{\pi}}\, n^{-3/2} 4^n \qquad \textit{for 3-connected maps;}$$

$$\frac{\sqrt{3}}{9\sqrt{\pi}}\, n^{-3/2} \left(\frac{27}{4}\right)^n \qquad \textit{for 2-connected maps;}$$

$$\frac{1}{\sqrt{\pi}}\, n^{-3/2} 12^n \qquad \textit{for arbitrary maps.}$$

*Proof.* The following estimate for the Fine numbers is well-known [3]:

$$F_n \sim \frac{4}{9\sqrt{\pi}} n^{-3/2} 4^n.$$

Since the number of self-dual 3-connected maps is equal to $F_{n-1}$, the first claim follows. An application of Stirling's estimate to the explicit formulas proves the second and the third claim. □

Observe the term $n^{-3/2}$ that appears in the estimates instead of the usual $n^{-5/2}$ that appears as a rule in the enumeration of planar maps [4]. This conforms to the fact that requiring some kind of symmetry for a map usually changes the subexponential term in the asymptotic estimate. On the other hand, the exponential growth (disregarding polynomial terms) of maps, 2-connected maps and 3-connected maps with $2n$ edges is, respectively, $12^{2n}$, $(27/4)^{2n}$ and $4^{2n}$. Hence, in all cases the number of self-dual maps with $2n$ edges is roughly the square root of the total number of maps of the same kind. As we are going to see, this can be interpreted in the sense that a self-dual map with $2n$ edges is essentially made of two equal 'pieces' with $n$ edges, suitably rotated.

In the paper we make a systematic use of a standard bijection between maps with $n$ edges and quadrangulations with $n$ faces. Given a map $M$ with vertices $B$ and faces $W$, the associated quadrangulation $Q$ has as vertices $B \cup W$ and edges connecting $b \in B$ and $w \in W$ if the face $w$ is incident with $b$ (as many edges as incidences between $w$ and $b$). The quadrangulation $Q$ is rooted from the root vertex of $M$ to the root face of $M$, and inherits a natural embedding from that of $M$. See an illustration in Figure 3, where vertices in $B$ and $W$ are drawn black and white. All quadrangulations are taken with a fixed 2-colouring in such a way that the root vertex is coloured black. Observe that $Q$ can have faces consisting of a double edge (a digon) and an isthmus inside it, or a digon with another digon inside. A face consisting of four distinct vertices is called a square.

The following is also well-known: $M$ is 2-connected if and only if $Q$ has no multiple edges; and $M$ is 3-connected if and only if $Q$ has no multiple edges and no separating quadrangles, that is, cycles of length four not bounding a face. Observe that the cut-vertex $x$ in Figure 3 gives rise to a double edge between $x$ and $f$, and that the 2-cut $\{y, z\}$ produces a separating quadrangle $ygzf$.

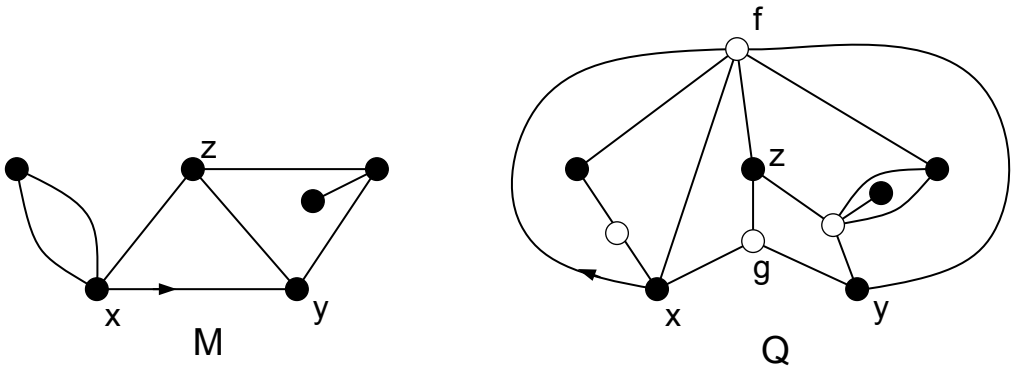


Figure 3: The quadrangulation associated to a map.

We now formulate a key observation, that allows us to work with quadrangulations having a certain symmetry instead of working directly self-dual maps. This simplifies the analysis greatly.

**Observation.** Let $M$ be a map, let $Q$ be the associated quadrangulation, and let $bw$ be the root edge. Then $M$ is self-dual if and only if the quadrangulation $Q'$ obtained from $Q$ by interchanging black and white colours and rooting it at edge $wb$ is isomorphic (as a rooted map) to $Q$. In plain words, $Q$ is the same when looked from the root edge $bw$ as from the root edge $wb$ after exchanging colours. We call such quadrangulations symmetric.

The rest of the paper is organised as follows. In Section 2 we enumerate 2-connected self-dual maps. Using the decomposition of a 2-connected map into 3-connected components (in terms of

quadrangulations), we enumerate 3-connected self-dual maps in Section 3. Using the decomposition of a connected map into 2-connected components (again in terms of quadrangulations), we enumerate connected self-dual maps in Section 4.

## 2 Two-connected maps

The enumeration of 2-connected self-dual maps is implicitly contained in the work of Brown [2]. In this section we show how to obtain item 2 from Theorem 1, and recall several results that are needed later.

From the previous observations, 2-connected self-dual maps with $2n$ edges are in one to one correspondence with symmetric quadrangulations $Q$ with $2n$ faces and without multiple edges. Throughout this section we only consider quadrangulations that have no multiple edges. Let $Q$ be a quadrangulation with at least 4 faces and root edge $b_1w_1$. Let $b_1w_1b_3w_2$ be the root face traversed in counter-clockwise order, and let $b_1w_1b_2w_3$ be the other face incident with $b_1w_1$. Then we redraw $Q$ as a quadrangulation $Q_h$ of a regular hexagon $b_1w_3b_2w_1b_3w_2$, except that the root edge is not drawn; see Figure 4.

**Lemma 3.** *With the previous notations, $Q$ is symmetric if and only if $Q_h$ is invariant under a rotation of $\pi$ degrees around the center of the hexagon.*

The proof is immediate from the definitions (note that we do not ask that the rotation preserves the colours of the vertices). Notice that symmetry of $Q$ is much easier to visualize in $Q_h$.

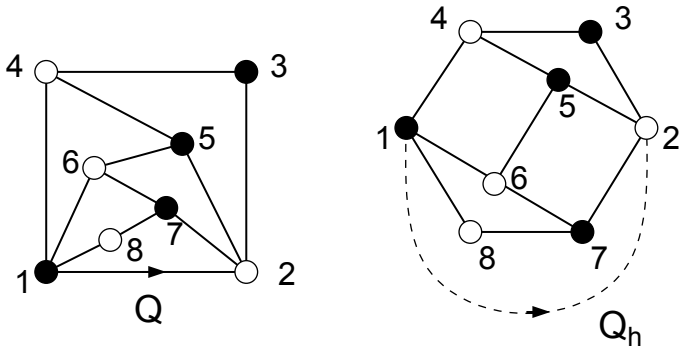


Figure 4: A symmetric quadrangulation and the associated quadrangulation of the hexagon.

Thus our goal is to count quadrangulations of $Q_h$ a hexagon with a rotational symmetry of $\pi$ degrees, but we are not allowed to use the edge $b_1w_1$ inside the hexagon, since this would produce a double edge in $Q$. For brevity, a quadrangulation (of a hexagon or otherwise) invariant by a rotation of $\pi$ degrees is called $\pi$-invariant.

The following result is contained in formula (11.12) from [2], with the value $p = 1$. The notation ${}_2U_{2n,2}$ there stands for quadrangulations with $2n$ inner vertices and $4+2 = 6$ outer vertices, having a rotational symmetry of order two. With these parameters, the number of inner faces is $2n + 2$.

**Lemma 4** (Brown)**.** *The number of $\pi$-invariant quadrangulations of a hexagon with $2n+2$ inner faces is equal to*

$$R_n = \frac{9(3n+2)!}{n!(2n+3)!}.$$

*Proof of item 2 from Theorem 1.* As we have just seen, the number of 2-connected self-dual planar maps with $2n$ edges equals the number of quadrangulations with $2n-2$ faces given in the previous lemma, minus those containing the edge $b_1w_1$. Of these there as many as arbitrary

quadrangulations with $n-1$ inner faces (take any quadrangulation below $b_1w_1$ and its symmetric above $b_1w_1$), and of these there as many as 2-connected maps with $n$ edges. Using formula (1) for $B_n$, the final result is

$$R_{n-2} - B_n = \frac{9(3n-4)!}{(n-2)!(2n-1)!} - \frac{2}{n(3n-2)}\binom{3n-2}{2n-1} = \frac{1}{n}\binom{3n-2}{n-1},$$

as was to be proved. □

In the rest of this section we state several formulas needed later on generating functions for several classes of quadrangulations. Let $B(z)$ be the generating function of quadrangulations of a square, according to the number of inner faces. The series $H^*(z)$ and $H(z)$ are generating functions for quadrangulations of a hexagon $b_1w_3b_2w_1b_3w_2$, in the first case arbitrary ones and in the second not containing the edge $b_1w_1$. The variable $z$ marks again the number of inner faces. The series $S_2^*(z)$ and $S_2(z)$ are the generating functions of those quadrangulations of the hexagon that are $\pi$-invariant, in the second case forbidding again the edge $b_1w_1$. This time $z$ marks half the number of number of inner faces (hence, $1+zS_2(z)$ is the generating function for 2-connected self-dual maps by half the number of edges).

The series $B(z)/z$ and $H^*(z)/z^2$ are the constant coefficient and the coefficient of $y^2$ in the series $U$ from [2, Eq. (6.3)], respectively; the series $S_2^*(z)/z$ is ${}_2U_{.2}$ from [2, Eq. (11.9)]. The expressions for these generating functions become quite compact thanks to a clever parametrisation due also to Brown. Let $u(z)$ be the solution to

$$u(z)(1-u(z))^2 = z.$$

Then

$$\begin{aligned}
B(z) &= \frac{(1-2u(z))u(z)}{(1-u(z))^2} = z + 2z^2 + 6z^3 + 22z^4 + 91z^5 + \cdots \\
H^*(z) &= \frac{(3-7u(z))u(z)^2}{(1-u(z))^3} = 3z^2 + 14z^3 + 63z^4 + 294z^5 + \cdots \\
H(z) &= H^*(z) - B(z)^2 = \frac{(2-6u(z)+3u(z)^2)u(z)^2}{(1-u(z))^4} = 2z^2 + 10z^3 + 47z^4 + 226z^5 + \cdots \\
S_2^*(z) &= \frac{3u(z)}{(1-u(z))} = 3z + 9z^2 + 36z^3 + 165z^4 + 819z^5 + \cdots \\
S_2(z) &= S_2^*(z) - B(z) = \frac{(2-u(z))u(z)}{(1-u(z))^2} = 2z + 7z^2 + 30z^3 + 143z^4 + 728z^5 + \cdots
\end{aligned} \tag{4}$$

Let us remark that, due to the bijection between maps and quadrangulations, the coefficient of $z^{n-1}$ in $B(z)$ is equal to $B_n$, as given in (1).

# 3 Three-connected maps

This section is devoted to the proof of the first item in Theorem 1. As in the previous section, we continue to work with the quadrangulation $Q_h$ of a hexagon obtained from the quadrangulation $Q$ associated to a map $M$. As mentioned above, $M$ is self-dual 3-connected if and only if $Q_h$ is $\pi$-invariant and together with the root $b_1w_1$ has no multiple edges or separating quadrangles. Therefore, we focus on enumerating such quadrangulations of a hexagon. In order to work with the quadrangulation $Q_h$ alone, we analyse how the root edge $b_1w_1$ may affect the conditions above. Let $b_1w_3b_2w_1b_3w_2$ be the boundary of the hexagon counter-clockwise. Then as in the 2-connected case, the edge $b_1w_1$ must not be present in the quadrangulation $Q_h$. Throughout this section, a quadrangulation of a hexagon has no multiple edges and does not contain the edge $b_1w_1$. As for separating quadrangles, we must forbid configurations that would create such a quadrangle with

the root edge. This happens if $Q_h$ contains one of the diagonal edges $b_2w_2$ and $b_3w_3$, or any path of the form $b_1xb_2, b_1xb_3$, $w_1xw_2$ or $w_1xw_3$, or a path of the form $b_1xyw_1$, where in all cases $x$ and $y$ are vertices not on the hexagon. We refer to these edges and paths as *forbidden paths*.

Let $Q_h$ be a $\pi$-invariant quadrangulation of a hexagon without separating quadrangles or forbidden paths. Each face $f$ of $Q_h$ is mapped to another face $f'$ of $Q_h$ by the $\pi$ degrees rotation, and since the number of faces is even, all faces come in pairs and there is no fixed face. For each pair of faces $\{f, f'\}$, choose a quadrangulation $q_f$ (which could be just a square) and replace both $f$ and $f'$ by $q_f$, in a way that the resulting quadrangulation $Q'$ of the hexagon is $\pi$-invariant. This process has created a quadrangulation $Q'$ with possibly some separating quadrangles. Also, observe that substituting a face by a quadrangulation cannot create new forbidden paths.

It is also easy to see that a $\pi$-invariant quadrangulation of a hexagon without forbidden paths can be obtained by the process above from a unique $\pi$-invariant quadrangulation of that hexagon without separating quadrangles or forbidden paths.

Next we translate this decomposition into generating functions. Let $S_3(z)$ be the generating function of $\pi$-invariant quadrangulations of a hexagon without separating quadrangles or forbidden paths, where $z$ marks half the number of inner faces (thus, $zS_3(z)$ is the generating function of 3-connected self-dual maps by half the number of edges). Let $P(z)$ be the generating function of $\pi$-invariant quadrangulations of a hexagon that contain some forbidden path, where again $z$ marks half the number of inner faces. Then

$$S_3(B(z)) = S_2(z) - P(z). \tag{5}$$

The following equation gives an expression for $P(z)$ in terms of $B(z), H(z)$ and $S_2(z)$.

$$P(z) = \frac{(1 + S_2(z))(H(z) + 2B(z))}{1 + H(z) + 2B(z)} \tag{6}$$

Proving this equation occupies the rest of this section, but before we show how to deduce from it the number of 3-connected self-dual maps, as stated in Theorem 1.

By replacing the right hand-side of Equation (5) by the expressions of $B(z), H(z)$ and $S_2(z)$ in terms of $u(z)$ given in (4), we arrive at

$$S_3(B(z)) = \frac{u(z)^2}{1 - 2u(z)}.$$

Now let $w(z)$ be the functional inverse of $B(z)$ and let $v(z) = u(w(z))$, so that

$$S_3(z) = v(z)^2/(1 - 2v(z)).$$

From the expression of $B(z)$ in terms of $u(z)$ we get

$$\frac{(1 - 2v(z))v(z)}{(1 - v(z))^2} = z,$$

and eliminating $v(z)$ from the last two equations we conclude that $S_3(z)$ is a root of

$$s^2(z^2 + 2z) + s(2z^2 + 2z - 1) + z^2.$$

We thus find

$$S_3(z) = \frac{1 - 2z - 2z^2 - \sqrt{1 - 4z}}{2z(2 + z)},$$

which equals $F(z) - 1$, where $F(z)$ is defined in (3).

We now turn to the proof of Equation (6). To find an expression for $P(z)$ we consider two classes $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\pi$-invariant quadrangulations of a hexagon; the class $\mathcal{E}_1$ includes all $\pi$-invariant quadrangulations that contain either an edge $b_2w_2$ or $b_3w_3$, or a path of the form $b_1xb_i$ or $w_1xw_i$

for $i \in \{2,3\}$; we call such a path a *short diagonal*. The class $\mathcal{E}_2$ is made of all $\pi$-invariant quadrangulations that contain a path $b_1xyw_1$, called a *long diagonal*. If $E_1(z)$ and $E_2(z)$ denote the associated generating functions and $E_{12}(z)$ is the generating function of $\mathcal{E}_1 \cap \mathcal{E}_2$ (the variable $z$ always marking half the number of inner faces) we have

$$P(z) = E_1(z) + E_2(z) - E_{12}(z).$$

We shall find (by dropping the variable for ease of reading):

$$E_1 = 2(1+S_2)\frac{B}{1+B}, \tag{7}$$

$$E_2 = \frac{H(1+S_2)}{1+H+2B}, \tag{8}$$

$$E_{12} = \frac{2B(B+H)(1+S_2)}{(1+B)(1+H+2B)}. \tag{9}$$

Then $E_1 + E_2 - E_{12}$ gives the value claimed in (6).

To find $E_1$ consider first a quadrangulation with a short diagonal $b_1xb_3$, and, moreover, among all possible choices for $x$, take the one that it is closest to $w_2$. Hence the quadrangulation restricted to the quadrangle $b_1xb_3w_2$ has no path of the form $b_1yb_3$; we call such a path a *black diagonal*. As the quadrangulation is $\pi$-invariant, there is a vertex $x'$ such that the square $w_1x'w_3b_2$ is the image of $b_1xb_3w_2$ by the $\pi$ degrees rotation. Therefore, a quadrangulation with a short diagonal $b_1xb_3$ can be decomposed as a $\pi$–invariant quadrangulation of an hexagon $b_1w_3x'w_1b_3x$ together with a quadrangulation of a square $b_1xb_3w_2$ without black diagonals and its image under a $\pi$ degrees rotation. (See the left hand-side of Figure 5.)

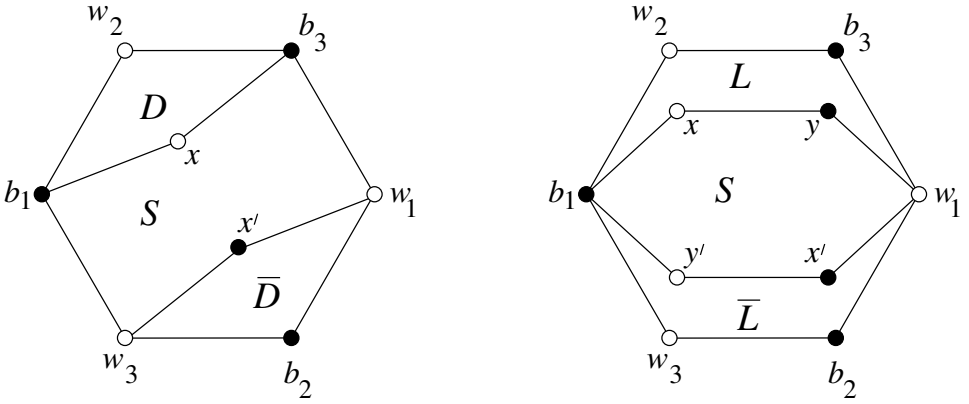


Figure 5: *Left.* A member of $\mathcal{E}_1$ with a short diagonal: $D$ is a quadrangulation without a black diagonal, $\bar{D}$ is the $\pi$ degrees rotation of $D$ and $S$ is a $\pi$-invariant quadrangulation. *Right.* A member of $\mathcal{E}_2$: $S$ is as before, $L$ is a quadrangulation without long diagonals or a short diagonal containing $x$ or $y$, and $\bar{L}$ is the $\pi$ degrees rotation of $L$.

The case of a diagonal of the form $b_1xb_2$ is identical, so that the only quadrangulations in the class $\mathcal{E}_1$ not considered so far are those that contain one of the edges $b_2w_2$ or $b_3w_3$ but no diagonal $b_1xb_i$ for $i \in \{2,3\}$. Those are easily decomposed as a quadrangulation of a square without black diagonals and its image under a $\pi$ degrees rotation. Thus, if $D$ is the generating function of quadrangulations of a square without black diagonals, we have $E_1 = 2(1+S_2)D$. To find $D$, note that an arbitrary quadrangulation of a square can be seen as a sequence of quadrangulations without black diagonals, so that $B = 1/(1-D) - 1$ and

$$D = \frac{B}{1+B}.$$

We have thus proved Equation (7).

We now analyse the class $\mathcal{E}_2$. Consider a $\pi$–invariant quadrangulation of a hexagon with a path of the form $b_1xyw_1$, and take $x$ and $y$ to be closest to the edge $w_2b_3$. If there are only one such $x$ and $y$, the path $b_1xyw_1$ splits the hexagon into two hexagons, whose quadrangulations are obtained from each other by a $\pi$ degrees rotation. If there is more than one path, then there are $x', y'$ such that the initial quadrangulation is split into three hexagons, the middle one being $\pi$-invariant and the other two being images of each other (see the right hand-side of Figure 5). Note that the quadrangulation of the hexagon $b_1xyw_1b_3w_2$ is not arbitrary, because by the choice of $x$ and $y$ it must not contain any path that gives rise to a long diagonal of the initial quadrangulation. Let $L$ be the generating function of such quadrangulations. Then

$$E_2 = L + S_2L.$$

We classify the quadrangulations of an hexagon $b_1xyw_1b_3w_2$ that do give rise to a long diagonal of the quadrangulation of $b_1w_3b_2w_1b_3w_2$ into two non-disjoint classes $\mathcal{L}_1$ and $\mathcal{L}_2$. The class $\mathcal{L}_1$ consists of those quadrangulations that contain a long diagonal and $\mathcal{L}_2$ of those that contain a short diagonal with one end in either $x$ or $y$. If $L_1(z)$ and $L_2(z)$ are the corresponding generating functions, and $L_{12}(z)$ is the generating function of $\mathcal{L}_1 \cap \mathcal{L}_2$ (with $z$ always marking the number of inner faces), we have

$$L = H - (L_1 + L_2 - L_{12}).$$

By further decomposing the quadrangulations it is easy to show that $L_1 = LH$, $L_2 = 2DH$ and $L_{12} = 2(DHL + DBL)$; we just note that the third case splits into two subcases, corresponding to whether the long and the short diagonal are disjoint or not. Adding these contributions together, we arrive at

$$L = \frac{H(1-2D)}{1 + H(1-2D) - 2DB},$$

which proves Equation (8).

Finally, we consider those quadrangulations that contain both short and long diagonals. As before, let $x$ be such that $b_1xb_3$ is a short diagonal and the quadrangulation restricted to $b_1xb_3w_2$ has no black diagonal. By the $\pi$ degrees rotation invariance, the initial quadrangulation decomposes as a quadrangulation without black diagonals of the square $b_1xb_3w_2$, together with its $\pi$ degrees rotation image, and a $\pi$-invariant quadrangulation of the hexagon $b_1w_3x'w_1b_3x$, such that the resulting quadrangulation has a long diagonal. This long diagonal can come from either the quadrangulation of this hexagon $b_1w_3x'w_1b_3x$ or from a short diagonal $w_1yx$ (implying the short diagonal $b_1y'x'$) or from the edge $xx'$. In terms of generating functions, all this translates in to

$$E_{12} = 2D\left(E_2 + \frac{E_1 - E_{12}}{2}\right).$$

Hence

$$E_{12} = \frac{DE_1 + 2DE_2}{1+D},$$

from which Equation (9) follows.

# 4 Arbitrary maps

To prove item 3 from Theorem 1 we need to enumerate symmetric quadrangulations with multiple edges allowed. Let $S_1(z)$ be the generating function of symmetric quadrangulations with multiple edges allowed, where $z$ marks half the number of faces. The outer face of such a quadrangulation can be a proper quadrangle (i.e., with no multiple edges), a pair of digons (two pairs of parallel edges) or a digon and a bridge. We treat the three cases separately.

In the case of the outer face being a digon and a bridge, it is easy to check that for the quadrangulation to be symmetric, the root edge must be one of the edges of the digon. As the other

face adjacent to the root edge must also be formed by a bridge and a digon, the quadrangulation is, up to reflection, of the form depicted in Figure 6, where the shaded area corresponds to a quadrangulation of a digon that is $\pi$-invariant (in a degenerate case, the shaded area would be reduced to just one edge).

There is a straightforward bijection between $\pi$-invariant quadrangulations of a digon and arbitrary symmetric quadrangulations: simply remove the edge of the digon that is parallel to the root edge. The contribution of this case to $S_1(z)$ is thus $2(z + zS_1(z))$.

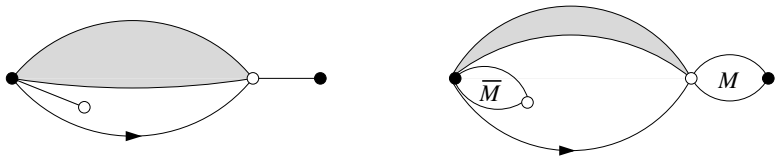


Figure 6: Possible structures of a symmetric quadrangulation when the outer face is not a square.

The case of the outer face being composed of two digons is similar; there are only two possibilities, and the only possibility, up to reflection, is depicted in Figure 6. The grey area is again a $\pi$-invariant quadrangulation of a digon (or just an edge), $S$ is an arbitrary quadrangulation of a digon and $\bar{S}$ is a copy of $S$ suitable placed such that resulting quadrangulation symmetric. As before, removing the edge parallel to the root in the outer face is a bijection between arbitrary quadrangulations of a digon and arbitrary quadrangulations. Then the contribution of this case to $S_1(z)$ is $2(M(z) - 1)(z + zS_1(z))$, where $M(z)$ is the generating function in Equation (2).

Finally, if the outer face is a square, removing the root edge leaves a map where the two faces that were adjacent to the root edge have merged into a face of length 6, and all other faces remain quadrangles. This face of length 6 can be a proper hexagon, as in previous sections, but can also be composed of a digon and two bridges, two digons and one bridge, three digons, or three bridges. We depict all possibilities in Figure 7, where as before a shaded area represents a $\pi$-invariant quadrangulation of the corresponding polygon and $S$ and $\bar{S}$ are two copies of the same quadrangulation of a digon.

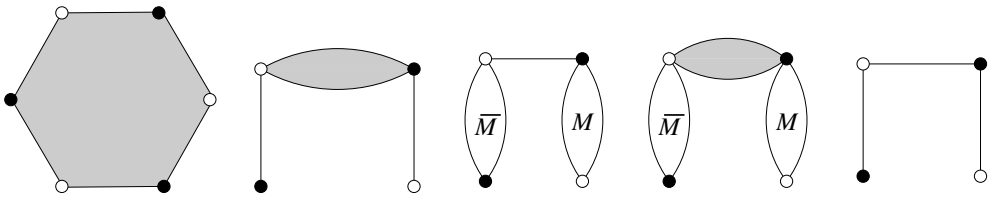


Figure 7: Possible structures of a symmetric quadrangulation of a square, with the root edge removed.

The contribution to $S_1(z)$ of all cases, except of the hexagon, follows as in the previous paragraphs, and equals $zS_1(z) + z(M(z) - 1) + zS_1(z)(M(z) - 1) + z = M(z)(z + zS_1(z))$.

Now we only need to count $\pi$-invariant quadrangulations of a hexagon that can have multiple edges. To construct such a quadrangulation, start from a quadrangulation $Q$ of a hexagon $b_1w_3b_3w_1b_2w_2$ without multiple edges and $\pi$-invariant (but allowing the presence of the edge $b_1w_1$). It follows that edges are paired by the $\pi$ degrees rotation, except for one which is mapped to itself by the rotation. In $Q$, replace this edge by a symmetric quadrangulation of a digon, and replace every edge and its image under the rotation by the same rotation of a digon (making the replacement so that the resulting quadrangulation of the hexagon is still $\pi$-invariant). This procedure gives all quadrangulations of a hexagon $\pi$-invariant, allowing multiple edges and the edge $b_1w_1$.

Taking into account that a quadrangulation of a hexagon with $2i$ inner faces has a total of $2(2i+1)+1$ edges, the contribution of this case is $1+(1+zS_1(z))(1+zM(z))S_2^*(z(1+zM(z))^2)$.

Adding everything together gives the equation

$$S_1(z) = M(z)(z + zS_1(z))(3 + S_2^*(zM(z)^2)). \tag{10}$$

From the expression of $S_2^*(z)$ in terms of $u(z)$ and the equation for $u(z)$, we obtain that $S = S_2^*(zM(z)^2)$ satisfies

$$27zM(z)^2 + (27zM(z)^2 - 9)S + 9zM(z)^2S^2 + zM(z)^2S^3 = 0.$$

With this equation we can eliminate $S_2^*(zM(z)^2)$ from Equation (10), and from the result eliminate $M(z)$ with Equation (2), obtaining finally the following equation for $S_1(z)$.

$$S_1(z) = 3z + 6zS_1(z) + 3zS_1(z)^2,$$

giving $S_1(z) = \sum_{n\geq 1} 3^n C(n)$, as claimed.

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