A Survey on Partially Ordered Patterns

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Abstract

The paper offers an overview over selected results in the literature on partially ordered patterns (POPs) in permutations, words and compositions. The POPs give rise in connection with co-unimodal patterns, peaks and valleys in permutations, Horse permutations, Catalan, Narayana, and Pell numbers, bi-colored set partitions, and other combinatorial objects.

Keywords: (partially ordered) pattern, POPs, non-overlapping occurrence, peak, valley, q-analogue, flat poset, co-unimodal pattern, generating function, distribution, bi-colored set partition, permutation, word, composition

1 Introduction

An occurrence of a *pattern* τ in a permutation π is defined as a subsequence in π (of the same length as τ) whose letters are in the same relative order as those in τ. For example, the permutation 31425 has three occurrences of the pattern 1-2-3, namely the subsequences 345, 145, and 125. *Generalized permutation patterns* (*GPs*) being introduced in [1] allow the requirement that some adjacent letters in a pattern must also be adjacent in the permutation. We indicate this requirement by removing a dash in the corresponding place. Say, if pattern 2-31 occurs in a permutation π, then the letters in π that correspond to 3 and 1 are adjacent. For example, the permutation 516423 has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563. Placing "[" on the left (resp., "]" on the right) next to a pattern p means the requirement that p must begin (resp., end) from the leftmost (resp., rightmost) letter. For example, the permutation 32415 contains two occurrences of the pattern [2-13, namely the subwords 324 and 315 and no occurrences of the pattern 3-2-1]. We refer to [5] and [20] for more information on patterns and GPs.

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A further generalization of the GPs (see [19]) is *partially ordered patterns* (*POPs*), where the letters of a pattern form a partially ordered set (poset), and an occurrence of such a pattern in a permutation is a linear extension of the corresponding poset in the order suggested by the pattern (we also pay attention to eventual dashes and brackets). For instance, if we have a poset on three elements labeled by $1'$, 1, and 2, in which the only relation is $1 < 2$ (see Figure 1), then in an occurrence of $p = 1'\text{-}12$ in a permutation π the letter corresponding to the $1'$ in p can be either larger or smaller than the letters corresponding to 12. Thus, the permutation 31254 has three occurrences of p, namely 3-12, 3-25, and 1-25.

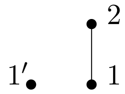

Figure 1: A poset on three elements with the only relation $1 < 2$.

Let $\mathcal{S}_n(p_1, \ldots, p_k)$ denote the set of n-permutations avoiding simultaneously each of the patterns $p_1, \ldots, p_k$.

The POPs were introduced in [17][1] as an auxiliary tool to study the maximum number of non-overlapping occurrences of *segmented* GPs (*SGPs*), also known as *consecutive* GPs, that is, the GPs, occurrences of which in permutations form contiguous subwords (there are no dashes). However, the most useful property of POPs known so far is their ability to "encode" certain sets of GPs which provides a convenient notation for those sets and often gives an idea how to treat them. For example, the original proof of the fact that $|\mathcal{S}_n(123, 132, 213)| = \binom{n}{\lfloor n/2 \rfloor}$ took 3 pages ([16]); on the other hand, if one notices that $|\mathcal{S}_n(123, 132, 213)| = |\mathcal{S}_n(11'2)|$, where the letters 1, $1'$, and 2 came from the same poset as above, then the result is easy to see. Indeed, we may use the property that the letters in odd and even positions of a "good" permutation do not affect each other because of the form of $11'2$. Thus we choose the letters in odd positions in $\binom{n}{\lfloor n/2 \rfloor}$ ways, and we must arrange them in decreasing order. We then must arrange the letters in even positions in decreasing order too.

The POPs can be used to encode certain combinatorial objects by restricted permutations. Examples of that are Theorem 1, Propositions 10, 11, 17, and 20, as well as several other results in the literature (see, e.g., [4]). Such encoding is interesting from the point of view of finding bijections between the sets of objects involved, but it also may have applications for enumerating certain statistics. The idea is to encode a set of objects under consideration as a set of permutations satisfying certain restrictions (given by certain POPs); under appropriate encodings, this allows us to transfer the interesting statistics from the original set to the set of permutations, where they are easy to handle. For an illustration of how encoding by POPs

[1]The POPs in this paper, as well as in [19], are the same as the POGPs in [17], which is an abbreviation for Partially Ordered Generalized Patterns.

can be used, see [22, Thm. 2.4] which deals with POPs in *compositions* (discussed in Section 5) rather than in permutations, though the approach remains the same.

As a matter of fact, some POPs appeared in the literature before they were actually introduced. Thus the notion of a POP allows us to collect under one roof (to provide a uniform notation for) several combinatorial structures such as *peaks*, *valleys*, *modified maxima* and *modified minima* in permutations, *Horse permutations* and *p-descents* in permutations discussed in Section 2.

There are several other ways to define occurrences of patterns in permutations (and other combinatorial objects like words and compositions) for which POPs can be defined and studied (see, e.g., [30] where certain POPs are studied in connection with *cyclic occurrence of patterns*). However, this survey deals with occurrences of patterns in the sense specified above.

The paper is organized as follows. Section 2 deals with *co-unimodal patterns* and some of their variations. In particular, this involves considering *peaks* and *valleys* in permutations, as well as so called *V- and* $\Lambda-$*patterns.* Sections 3 and 4 discuss POPs with, and without, dashes involved, respectively. In particular, Section 3 deals with *Horse permutations* and *multi-patterns*, while Section 4 presents results on *flat posets*, *non-overlapping SPOPs* in permutations and words, and *q-analogues* for non-overlapping SPOPs (*SPOP* abbreviates Segmented POP). Further, Section 5 states some of results on POPs in *compositions*, which can be viewed as a generalization for certain results on POPs in words. Finally, in Section 6, we state a couple of concluding remarks.

In what follows we need the following notations. Let σ and τ be two POPs of length greater than 0. We write $\sigma < \tau$ to indicate that any letter of σ is less than any letter of τ. We write $\sigma <> \tau$ when no letter in σ is comparable with any letter in τ. The *GF* (*EGF*; *BGF*) denotes the (*exponential*; *bivariate*) *generating function.* If $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n$, then the *reverse* of π is $\pi^r := a_n\cdots a_2a_1$, and the *complement* of π is a permutation π^c such that $\pi^c_i = n+1-a_i$, where $i \in [n] = \{1,\ldots,n\}$. We call π^r, π^c, and $(\pi^r)^c = (\pi^c)^r$ *trivial bijections.*

2 Co-unimodal patterns and their variations

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, the *inversion index*, $\mathrm{inv}(\pi)$, is the number of ordered pairs (i,j) such that $1 \leq i < j \leq n$ and $\pi_i > \pi_j$. The major index, $\mathrm{maj}(\pi)$, is the sum of all i such that $\pi_i > \pi_{i+1}$. Suppose σ is a SPOP and

$$\mathrm{place}_\sigma(\pi) = \{i \mid \pi \text{ has an occurrence of } \sigma \text{ starting at } \pi_i\}.$$

Let $\mathrm{maj}_\sigma(\pi)$ be the sum of the elements of $\mathrm{place}_\sigma(\pi)$.

If σ is *co-unimodal*, meaning that $k = \sigma_1 > \sigma_2 > \cdots > \sigma_j < \cdots < \sigma_k$ for some $2 \leq j \leq k$ (see Figure 2 for a corresponding poset in the case $j = 3$

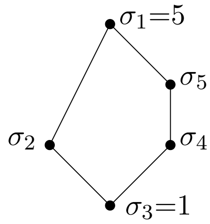

Figure 2: A poset for co-unimodal pattern in the case $j = 3$ and $k = 5$.

and $k = 5$), then the following formula holds [3]:

$$\sum_{\pi \in \mathcal{S}_n} t^{\mathrm{maj}_\sigma(\pi^{-1})} q^{\mathrm{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{maj}_\sigma(\pi^{-1})} q^{\mathrm{inv}(\pi)}.$$

If $k = 2$ we deal with usual descents. Thus a co-unimodal pattern can be viewed as a generalization of the notion of a descent. This may be a reason why a co-unimodal pattern p is called *p-descent* in [3]. Also, setting $t = 1$ we get a well-known result by MacMahon on equidistribution of maj and inv.

The notion of co-unimodal patterns was refined and generalized in [29], where the authors use symmetric functions along with *λ-brick tabloids* and *weighted λ-brick tabloids* to obtain their (new) results as well as some known results. Moreover, in all the cases in [29], it is possible to extend the results to q-analogues, where the powers of q count the inversion statistic. See [28] for basic techniques and ideas used in [29].

2.1 Peaks and valleys in permutations

A permutation π has exactly k *peaks* (resp., *valleys*), also known as *maxima* (resp., *minima*), if $|\{j \mid \pi_j > \max\{\pi_{j-1}, \pi_{j+1}\}\}| = k$ (resp., $|\{j \mid \pi_j < \min\{\pi_{j-1}, \pi_{j+1}\}\}| = k$). Thus, an occurrence of a peak in a permutation is an occurrence of the SPOP $1'21''$, where relations in the poset are $1' < 2$ and $1'' < 2$. Similarly, occurrences of valleys correspond to occurrences of the SPOP $2'12''$, where $2' > 1$ and $2'' > 1$. See Figure 3 for the posets corresponding to the peaks and valleys. So, any research done on the peak (or valley) statistics can be regarded as research on (S)POPs (e.g., see [33]).

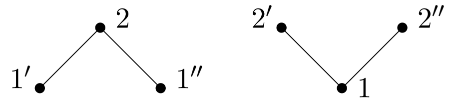

Figure 3: Posets corresponding to peaks and valleys.

Also, results related to *modified maxima* and *modified minima* can be viewed as results on SPOPs. For a permutation $\sigma_1 \dots \sigma_n$ we say that σ_i is a *modified maximum* if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$ and a *modified minimum* if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$, for $i = 1, \dots, n$, where $\sigma_0 = \sigma_{n+1} = 0$. Indeed, we can view a pattern p as a function from the set of all symmetric groups $\cup_{n \geq 0} \mathcal{S}_n$ to the set of natural numbers such that $p(\pi)$ is the number of occurrences of

p in π, where π is a permutation. Thus, studying the distribution of modified maxima (resp., minima) is the same as studying the function $ab] + 1'21'' + [dc$ (resp., $ba] + 2'12'' + [cd$) where $a < b$, $c < d$ and the other relations between the patterns' letters are taken from Figure 3. Also, recall that placing "[" (resp., "]") next to a pattern p means the requirement that p must begin (resp., end) with the leftmost (resp., rightmost) letter.

A specific result in this direction is problem 3.3.46(c) on page 195 in [11]: We say that σ_i is a *double rise* (resp., *double fall*) if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$ (resp., $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$); The number of permutations in $\mathcal{S}_n$ with i_1 modified minima, i_2 modified maxima, i_3 double rises, and i_4 double falls is

$$\left[u_1^{i_1}u_2^{i_2-1}u_3^{i_3}u_4^{i_4}\frac{x^n}{n!}\right]\frac{e^{\alpha_2 x}-e^{\alpha_1 x}}{\alpha_2 e^{\alpha_1 x}-\alpha_1 e^{\alpha_2 x}}$$

where $\alpha_1\alpha_2 = u_1u_2$, $\alpha_1+\alpha_2 = u_3+u_4$.

In Corollary 30 one has an explicit generating function for the distribution of peaks (valleys) in permutations. This result is an analogue to a result in [9] where the circular case of permutations is considered, that is, when the first letter of a permutation is thought to be to the right of the last letter in the permutation. In [9] it is shown that if $M(n,k)$ denotes the number of circular permutations in $\mathcal{S}_n$ having k maxima, then

$$\sum_{n\geq 1}\sum_{k\geq 0} M(n,k)y^k\frac{x^n}{n!} = \frac{zx(1-z\tanh xz)}{z-\tanh xz}$$

where $z = \sqrt{1-y}$.

2.2 V- and $\Lambda-$patterns

A variation of co-unimodal patterns is when we do not require in a co-unimodal pattern the first element to be the largest one. More precisely, we say that a factor $\pi_{i-k}\cdots\pi_i\cdots\pi_{i+\ell}$ of a permutation $\pi_1\cdots\pi_n$ is an occurrence of the pattern $V(k,\ell)$ (resp. $\Lambda(k,\ell)$) if $\pi_{i-k} > \pi_{i-k+1} > \cdots > \pi_i < \pi_{i+1} < \cdots < \pi_{i+\ell}$ (resp. $\pi_{i-k} < \pi_{i-k+1} < \cdots < \pi_i > \pi_{i+1} > \cdots > \pi_{i+\ell}$). Such patterns are a refinement of the concept of peaks and valleys.

A general approach to study avoidance of V- and $\Lambda-$patterns is suggested in [23] (see [23, Subsec. 2.2]). Below, we list explicit enumerative results in [23] starting with the one having a combinatorial interpretation for avoidance of a certain V-pattern.

Let K_n' denote the *corona* of the complete graph K_n and the complete graph K_1; in other words, K_n' is the graph constructed from K_n by adding for each vertex v a new vertex v' and the edge vv'. The following theorem provides a combinatorial property involving the pattern $V(1,2)$.

Theorem 1. ([23, Thm 7]) *The set of $(n+1)$-permutations avoiding simultaneously the patterns 213 and $V(1,2)$ is in one-to-one correspondence with* *the set of all matchings of K'_n. Thus, the EGF for the number of permutations avoiding the patterns 213 and $V(1,2)$ is given by*

$$A(x) = 1 + \int_0^x e^{2t+\frac{t^2}{2}}\, dt.$$

Theorem 2. ([23, Thm 1]) *The EGF $A(x)$ for the number of permutations avoiding $V(2,1)$ is given by*

$$1 + \exp\left(\frac{3x}{2}\right)\sec\left(\frac{\sqrt{3}x}{2}+\frac{\pi}{6}\right)\int_0^x \exp\left(-\frac{3u}{2}\right)\cos\left(\frac{\sqrt{3}u}{2}+\frac{\pi}{6}\right)du.$$

Theorem 3. ([23, Thm 2]) *The EGF $A(x)$ for the number of permutations avoiding simultaneously the patterns $V(1,2)$, $V(2,1)$, and $\Lambda(1,2)$ is given by*

$$\frac{1}{2}(e^x + (\tan x + \sec x)(e^x+1) - (1+2x+x^2)).$$

Theorem 4. ([23, Thm 2]) *The EGF $A(x)$ for the number of permutations avoiding simultaneously the patterns $V(1,2)$ and $\Lambda(1,2)$ is given by*

$$1 + x + (\tan x + \sec x - 1)(e^x - 1).$$

Theorem 5. ([23, Cor 5]) *The EGF $A(x)$ for the number of permutations avoiding simultaneously the patterns $V(1,2)$ and $\Lambda(2,1)$ is given by*

$$\frac{\sqrt{3}}{2}\exp\left(\frac{x}{2}\right)\sec\left(\frac{\sqrt{3}}{2}x+\frac{\pi}{6}\right) + e^x - \left(1+x+\frac{x^2}{2}\right).$$

Theorem 6. ([23, Thm 6]) *The number of n-permutations avoiding $V(1,2)$ and $V(2,1)$ simultaneously is given by*

$$A_n = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ n-i-j \text{ is odd}}}^{n-i+1} A^n_{i,j}$$

with

$$A^n_{i,j} = \begin{cases} \binom{n-1}{i-1} & \text{if } n \geq i \geq 1 \text{ and } \\ & n-i-j=-1, \\ \binom{n}{i}\binom{n-i}{j} - \binom{n-1}{i-1} - \binom{n-1}{i} - \binom{n-1}{i+1} & \text{if } n-i-j=1, \\ \binom{n}{i}\binom{n-i}{j}E_{n-i-j} - A^n_{i+2,j} - A^n_{i,j+2} - A^n_{i+2,j+2} & \text{if } n-i-j \geq 3 \text{ is odd} \end{cases}$$

where E_n is the number of alternating permutations.

3 POPs involving dashes

In this section we consider some of the results on POPs involving at least one dash.

3.1 Patterns containing □-symbol

In [15] the authors study simultaneous avoidance of the patterns 1-3-2 and 1□23. A permutation π avoids 1□23 if there is no $\pi_i < \pi_j < \pi_{j+1}$ with $i < j-1$. Thus the □ symbol has the same meaning as "-" except for □ does not allow the letters separated by it to be adjacent in an occurrence of the corresponding pattern. In the POP-terminology, 1□23 is the pattern 1-1′-23, or 1-1′23, or 11′-23, where 1′ is incomparable with the letters $1, 2$, and 3 which, in turn, are ordered naturally: $1 < 2 < 3$. The permutations avoiding 1-3-2 and 1□23 are called *Horse permutations*. The reason for the name came from the fact that these permutations are in one to one correspondence with *Horse paths*, which are the lattice paths from (0,0) to (n, n) containing the steps $(0, 1)$, $(1, 1)$, $(2, 1)$, and $(1, 2)$ and not passing the line $y = x$. According to [15], the generating function for the horse permutations is

$$\frac{1-x-\sqrt{1-2x-3x^2-4x^3}}{2x^2(1+x)}.$$

Moreover, in [15] the generating functions for Horse permutations avoiding, or containing (exactly) once, certain patterns are given.

In [10], patterns of the form x-y□z are studied, where $xyz \in \mathcal{S}_3$. Such a pattern can be written in the POP-notation as, for example, x-y-a-z where a is not comparable to x, y, and z. A bijection between permutations avoiding the pattern 1-2□3, or 2-1□3, and the set of *odd-dissection convex polygons* is given. Moreover, generating functions for permutations avoiding 1-3□2 and certain additional patterns are obtained in [10].

3.2 Patterns of the form σ-m-τ

Let σ and τ be two SGPs (the results below work for SPOPs as well). We consider the POP $\alpha = \sigma$-m-τ with $m > \sigma$, $m > \tau$, and $\sigma <> \tau$, that is, each letter of σ is incomparable with any letter of τ and m is the largest letter in α. The POP α is an instance of so called *shuffle patterns* (see [17, Sec 4]).

Theorem 7. ([17, Thm. 16]) *Let $A(x)$, $B(x)$ and $C(x)$ be the EGF for the number of permutations that avoid σ, τ and α respectively. Then $C(x)$ is the solution to the following differential equation with $C(0) = 1$:*

$$C'(x) = (A(x) + B(x))C(x) - A(x)B(x).$$

If τ is the empty word then $B(x) = 0$ and we get the following result for segmented GPs:

Corollary 8. ([17, Thm. 13],[24]) *Let $\alpha = \sigma$-m, where σ is a SGP on $[k-1]$. Let $A(x)$ (resp., $C(x)$) be the EGF for the number of permutations that avoid σ (resp., α). Then $C(x) = e^{F(x,A)}$, where $F(x, A) = \int_0^x A(y)\, dy$.*

Example 1. ([17, Ex 15]) Suppose $\alpha = 12$-3. Here $\sigma = 12$, whence $A(x) = e^x$, since there is only one permutation that avoids σ. So

$$C(x) = e^{F(x,\exp)} = e^{e^x - 1}.$$

We get [6, Prop. 4] since $C(x)$ is the EGF for the Bell numbers.

Corollary 9. ([17, Cor. 19]) *Let $\alpha = \sigma\text{-}m\text{-}\tau$ is as described above. We consider the pattern $\varphi(\alpha) = \varphi_1(\sigma)\text{-}m\text{-}\varphi_2(\tau)$, where φ_1 and φ_2 are any trivial bijections. Then $|\mathcal{S}_n(\alpha)| = |\mathcal{S}_n(\varphi(\alpha))|$.*

3.3 Patterns of the form m-σ-m

This subsection contains results on patterns in which two largest *incomparable* elements of the corresponding poset embrace the other elements building a consecutive POP (SPOP).

Proposition 10. ([12]) *Suppose the elements $1, 2, 3', 3''$ build the poset with the relations $1 < 2 < 3'$ and $2 < 3''$ ($3'$ is incomparable with $3''$). Then permutations avoiding the POP $3'$-12-$3''$ are in one-to-one correspondence with* bi-colored set partitions.

Proposition 11. ([12]) *Suppose the elements $1', 1'', 2, 3', 3''$ build the poset with the relations $1', 1'' < 2 < 3', 3''$ ($1'$ is incomparable with $1''$, and $3'$ is incomparable with $3''$). Then permutations avoiding the POP $3'$-$1'21''$-$3''$ are in one-to-one correspondence with* Dowling partitions. *Moreover, the EGF for such permutations is given by*

$$1 + \int_0^x \exp\left(\frac{e^t + 2t - 1}{2}\right)\, dt.$$

3.4 Multi-patterns

Suppose $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is a set of segmented GPs and $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ where each letter of σ_i is incomparable with any letter of σ_j whenever $i \neq j$ ($\sigma_i <> \sigma_j$). We call such POPs *multi-patterns*. Clearly, the Hasse diagram for such a pattern is k disjoint chains similar to that in Figure 4.

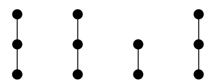

Figure 4: A poset corresponding to a multi-pattern.

Theorem 12. ([17, Thm. 23 and Cor. 24]) *The number of permutations avoiding the pattern $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ is equal to that avoiding a multi-pattern obtained from p by an arbitrary permutation of σ_i's as well as by applying to σ_i's any of trivial bijections.*

The following theorem is the basis for calculating the number of permutations that avoid a multi-pattern.

Theorem 13. ([17, Thm. 28]) *Let* $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ *be a multi-pattern and let* $A_i(x)$ *be the EGF for the number of permutations that avoid* σ_i*. Then the EGF* $A(x)$ *for the number of permutations that avoid* p *is*

$$A(x) = \sum_{i=1}^{k} A_i(x) \prod_{j=1}^{i-1} ((x-1)A_j(x) + 1).$$

Corollary 14. ([17, Cor. 26]) *Let* $p = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ *be a multi-pattern, where* $|\sigma_i| = 2$ *for all* i*. That is, each* σ_i *is either 12 or 21. Then the EGF for the number of permutations that avoid* p *is given by*

$$A(x) = \frac{1 - (1 + (x-1)e^x)^k}{1-x}.$$

Remark 15. Although the results in Theorems 12 and 13 are stated in [17] for σ_i's which are SGPs, they are true for σ_is which are SPOPs ([19, Remark 7]).

4 Segmented POPs (SPOPs)

Patterns in Section 2 are also examples of SPOPs. In fact, the most of known results on POPs are related to SPOPs.

4.1 Segmented patterns of length four

In this subsection we provide the known results related to SPOPs of length four. Theorem 2, Proposition 24, and Corollary 29 give extra results on such patterns. In this subsection, $A(x) = \sum_{n\geq 0} A_n x^n/n!$ is the EGF for the number of permutations in question. The patterns in the subsection are built on the poset in Figure 6 and the letter $1''$ is not comparable to any other letter.

Theorem 16. ([17, Thm. 30]) *For the SPOP* $122'1'$*, we have that*

$$A(x) = \frac{1}{2} + \frac{1}{4}\tan x(1 + e^{2x} + 2e^x \sin x) + \frac{1}{2}e^x \cos x.$$

Proposition 17. ([18, Prop. 8,9]) *There are* $\binom{n-1}{\lfloor (n-1)/2\rfloor}\binom{n}{\lfloor n/2\rfloor}$ *permutations in* $\mathcal{S}_n$ *that avoid the SPOP* $12'21'$*. The* $(n+1)$*-permutations avoiding* $12'21'$ *are in one-to-one correspondence with different walks of* n *steps between lattice points, each in a direction N, S, E or W, starting from the origin and remaining in the positive quadrant.*

Proposition 18. ([18, Prop. 4,5,6]) *For the SPOP* $11'1''2$*, one has*

$$A_n = \frac{n!}{\lfloor n/3\rfloor!\lfloor (n+1)/3\rfloor!\lfloor (n+2)/3\rfloor!},$$

and for the SPOP $11'21''$ *and* $n \geq 1$*, we have* $A_n = n \cdot \binom{n-1}{\lfloor (n-1)/2\rfloor}$*. Moreover, for the SPOPs* $1'1''12$ *and* $1'121''$*, we have* $A_0 = A_1 = 1$*, and, for* $n \geq 2$*,* $A_n = n(n-1)$*.*

Proposition 19. ([18, Prop. 7]) *For the SPOP* $1231'$*, we have*

$$A(x) = xe^{x/2}\left(\cos\frac{\sqrt{3}x}{2} - \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}x}{2}\right)^{-1} + 1,$$

and for the SPOPs $1321'$ *and* $2131'$*, we have*

$$A(x) = x(1 - \int_0^x e^{-t^2/2}\,dt)^{-1} + 1.$$

We end up this subsection with a result on multi-avoidance of SPOPs that has a combinatorial interpretation.

Proposition 20. ([4, Prop. 2.1,2.2]) *There are* $2\binom{n}{\lfloor n/2\rfloor}$ *n-permutations avoiding the SPOPs* $11'22'$ *and* $22'11'$ *simultaneously. For* $n \geq 3$*, there is a bijection between such n-permutations and the set of all* $(n+1)$*-step walks on the x-axis with the steps* $a = (1,0)$ *and* $\bar{a} = (-1,0)$ *starting from the origin but not returning to it.*

4.2 SPOPs built on flat posets

In this subsection, we consider flat posets built on $k+1$ elements $a, a_1, \ldots, a_k$ with the only relations $a < a_i$ for all i. A Hasse diagram for the flat poset is in Figure 5. Theorem 26 and Corollary 30 are the main results in the subsection.

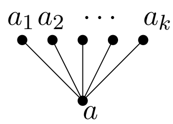

Figure 5: A flat poset.

The following proposition generalizes [6, Prop. 6]. Indeed, letting $k = 2$ in the proposition we deal with involutions and permutations avoiding 1-23 and 1-32. Note that even though Proposition 21 and Corollary 22 contain dashes in the patterns, those results are actually on SPOPs due to Proposition 23. (We stated the results with dashes to be consistent with [6, Prop. 6].)

Proposition 21. ([19, Prop. 14]) *The permutations in* $\mathcal{S}_n$ *having cycles of length at most* k *are in one-to-one correspondence with permutations in* $\mathcal{S}_n$ *that avoid* $a\text{-}a_1\cdots a_k$.

Corollary 22. ([19, Cor. 15]) *The EGF for the number of permutations avoiding* $a\text{-}a_1\cdots a_k$ *is given by* $\exp(\sum_{i=1}^k x^i/i)$.

Proposition 23. ([19, Prop. 16]) *One has* $\mathcal{S}_n(a\text{-}a_1\cdots a_k) = \mathcal{S}_n(aa_1\cdots a_k)$*, and thus the EGF for the number of permutations avoiding* $aa_1\cdots a_k$ *is* $\exp(\sum_{i=1}^k x^i/i)$.

Proposition 24. ([19, Cor. 17]) *The EGF for the number of permutations avoiding $aa_1a_2a_3$ is given by* $\exp(x + x^2/2 + x^3/3)$.

Theorem 25. (Distribution of $aa_1a_2\cdots a_k$, [19, Thm. 18]) *Let*

$$P := P(x,y) = \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n} y^{e(\pi)}x^n/n!$$

be the BGF on permutations, where $e(\pi)$ *is the number of occurrences of the SPOP* $p = aa_1a_2\cdots a_k$ *in* π. *Then* P *is the solution to*

$$\frac{\partial P}{\partial x} = yP^2 + \frac{(1-y)(1-x^k)}{1-x}P \tag{1}$$

with the initial condition $P(0,y) = 1$.

Note, that if $y = 0$ in Theorem 25, then the function in Corollary 22, due to Proposition 23, is supposed to be the solution to (1), which is true. If $k = 1$ in Theorem 25, then as the solution to (1) we get nothing else but the distribution of descents in permutations: $(1-y)(e^{(y-1)x} - y)^{-1}$. Thus Theorem 25 can be thought as a generalization of the result on the descent distribution.

The following theorem generalizes Theorem 25. Indeed, Theorem 25 is obtained from Theorem 26 by plugging in $\ell = 0$ and observing that obviously $aa_1\cdots a_k$ and $a_1\cdots a_ka$ are equidistributed.

Theorem 26. (Distribution of $a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$, [19, Thm. 19])) *Let*

$$P := P(x,y) = \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n} y^{e(\pi)}x^n/n!$$

be the BGF of permutations where $e(\pi)$ *is the number of occurrences of the SPOP* $p = a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ *in* π. *Then* P *is the solution to*

$$\frac{\partial P}{\partial x} = y\left(P - \frac{1-x^k}{1-x}\right)\left(P - \frac{1-x^\ell}{1-x}\right) + \frac{2-x^k-x^\ell}{1-x}P - \frac{1-x^k-x^\ell+x^{k+\ell}}{(1-x)^2}. \tag{2}$$

with the initial condition $P(0,y) = 1$.

If $y = 0$ in Theorem 26 then we get the following corollary:

Corollary 27. ([19, Cor. 20]) *The EGF* $A(x) = \sum_{n\geq 0} A_nx^n/n!$ *for the number of permutations avoiding the SPOP* $p = a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ *satisfies the following differential equation with the initial condition* $A(0) = 1$*:*

$$A'(x) = \frac{2-x^k-x^\ell}{1-x}A(x) - \frac{1-x^k-x^\ell+x^{k+\ell}}{(1-x)^2}.$$

The following corollaries to Corollary 27 are obtained by plugging in $k = \ell = 1$ and $k = 1$ and $\ell = 2$ respectively.

Corollary 28. ([16]) The EGF for the number of permutations avoiding a_1aa_2 is $(\exp(2x)+1)/2$ and thus $|\mathcal{S}_n(a_1aa_2)| = 2^{n-1}$.

Corollary 29. ([19, Cor. 22]) *The EGF for the number of permutations avoiding $a_1aa_2a_3$ is*

$$1+\sqrt{\frac{\pi}{2}}(\mathit{erf}(\frac{1}{\sqrt{2}}x+\sqrt{2})-\mathit{erf}(\sqrt{2}))e^{\frac{1}{2}x(x+4)+2}$$

where $\mathit{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ *is the error function.*

If $k=1$ and $\ell=1$, then our pattern a_1aa_2 is nothing else but the valley statistic. In [31] a recursive formula for the generating function of permutations with exactly k valleys is obtained, which however does not seem to allow (at least easily) finding the corresponding BGF. As a corollary to Theorem 26 we get the following BGF by solving (2) for $k=1$ and $\ell=1$:

Corollary 30. ([19, Cor. 23]) *The BGF for the distribution of peaks (valleys) in permutations is given by*

$$1-\frac{1}{y}+\frac{1}{y}\sqrt{y-1}\cdot\tan\left(x\sqrt{y-1}+\arctan\left(\frac{1}{\sqrt{y-1}}\right)\right).$$

4.3 Distribution of SPOPs on flat posets with additional restrictions

The results from this subsection are in a similar direction as that in the papers [2], [25], [26], and several other papers, where the authors study 1-3-2-avoiding permutations with respect to avoidance/counting of other patterns. Such a study not only gives interesting enumerative results, but also provides a number of applications (see [2]).

To state the theorem below, we define $P_k=\sum_{n=0}^{k-1}\frac{1}{n+1}\binom{2n}{n}x^n$. That is, P_k is the sum of initial k terms in the expansion of the generating function $\frac{1-\sqrt{1-4x}}{2x}$ of the Catalan numbers.

Theorem 31. (Distribution of $a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ on $\mathcal{S}_n$(2-1-3), [19, Thm. 24]) *Let*

$$P:=P(x,y)=\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3)}y^{e(\pi)}x^n$$

be the BGF of 2-1-3-avoiding permutations where $e(\pi)$ is the number of occurrences of the SPOP $p=a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$ in π. Then P is given by

$$\frac{1-x(1-y)(P_k+P_\ell)-\sqrt{(x(1-y)(P_k+P_\ell)-1)^2-4xy(x(y-1)P_kP_\ell+1)}}{2xy}.$$

We now discuss several corollaries to Theorem 31. Note that letting $y = 1$, we obtain the GF for the Catalan numbers. Also, letting $y = 0$ in the expansion of P, we obtain the GF for the number of permutations avoiding simultaneously the patterns 2-1-3 and $a_1a_2\cdots a_kaa_{k+1}a_{k+2}\cdots a_{k+\ell}$.

If $k = 1$ and $\ell = 0$ in Theorem 31, then $P_k = 1$ and $P_\ell = 0$, and we obtain the distribution of descents in 2-1-3-avoiding permutations. This distribution gives the *triangle of Narayana numbers* (see [32, A001263]).

If $k = \ell = 1$ in Theorem 31, then we deal with avoiding the pattern 2-1-3 and counting occurrences of the pattern 312, since any occurrence of a_1aa_2 in a legal permutation must be an occurrence of 312 and vice versa. Thus the BGF of 2-1-3-avoiding permutations with a prescribed number of occurrences of 312 is given by

$$\frac{1 - 2x(1-y) - \sqrt{4(1-y)x^2 + 1 - 4x}}{2xy}.$$

Reading off the coefficients of the terms involving only x in the expansion of the function above, we can see that the number of n-permutations avoiding simultaneously the patterns 2-1-3 and 312 is 2^{n-1}, which is known and is easy to see directly from the structure of such permutations.

Reading off the coefficients of the terms involving y to the power 1, we see that the number of n-permutations avoiding 2-1-3 and having exactly one occurrence of the pattern 312 is given by $(n-1)(n-2)2^{n-4}$. The corresponding sequence appears as [32, A001788] and it gives an interesting fact having a combinatorial proof:

Proposition 32. ([19, Prop. 25]) *There is a bijection between 2-dimensional faces in the $(n+1)$-dimensional hypercube and the set of 2-1-3-avoiding $(n+2)$-permutations with exactly one occurrence of the pattern 312.*

If $k = 1$ and $\ell = 2$ in Theorem 31, then we deal with avoiding the pattern 2-1-3 and counting occurrences of the pattern $a_1aa_2a_3$. In particular, one can see that the number of permutations avoiding simultaneously 2-1-3 and $a_1aa_2a_3$ is given by the *Pell numbers* $p(n)$ defined as $p(n) = 2p(n-1) + p(n-2)$ for $n > 1$; $p(0) = 0$ and $p(1) = 1$. The Pell numbers appear as [32, A000129], where one can find objects related to our restricted permutations.

4.4 Non-overlapping SPOPs

This subsection deals additionally with occurrences of patterns in words. The letters $1, 2, 1', 2'$ appearing in the examples below are ordered as in Figure 6.

Theorem 13 and its counterpart in the case of words [21, Thm. 4.3] and [21, Cor. 4.4], as well as Remark 15 applied for these results, give an interesting application of the multi-patterns in finding a certain statistic, namely the *maximum number of non-overlapping occurrences of a SPOP* in permutations and words. For instance, the maximum number of non-overlapping occurrences of the SPOP $11'2$ in the permutation 621394785 is

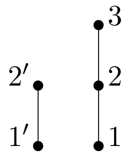

Figure 6: A poset giving partial order for 1,2,3,1′, and 2′.

2, and this is given by the occurrences 213 and 478, or the occurrences 139 and 478.

Theorem 33 generalizes [17, Thm. 32] and [21, Thm. 5.1].

Theorem 33. ([18, Thm. 16]) *Let p be a SPOP and $B(x)$ (resp., $B(x;k)$) is the EGF (resp., GF) for the number of permutations (resp., words over $[k]$) avoiding p. Let $D(x,y)=\sum_{\pi} y^{N_p(\pi)}\frac{x^{|\pi|}}{|\pi|!}$ and $D(x,y;k)=\sum_{n\geq 0}\sum_{w\in[k]^n} y^{N(w)}x^n$ where $N_p(s)$ is the maximum number of non-overlapping occurrences of p in s. Then $D(x,y)$ and $D(x,y;k)$ are given by*

$$\frac{B(x)}{1-y(1+(x-1)B(x))} \quad \textit{and} \quad \frac{B(x;k)}{1-y(1+(kx-1)B(x;k))}.$$

The following examples are corollaries to Theorem 33.

Example 2. ([18, Ex 1]) If we consider the SPOP $11'$ then clearly $B(x)=1+x$ and $B(x;k)=1+kx$. Hence,

$$D(x,y)=\frac{1+x}{1-yx^2}=\sum_{i\geq 0}(x^{2i}+x^{2i+1})y^i,$$

and

$$D(x,y;k)=\frac{1+kx}{1-y(kx)^2}=\sum_{i\geq 0}((kx)^{2i}+(kx)^{2i+1})y^i.$$

Example 3. ([18, Ex 2]) For permutations, the distribution of the maximum number of non-overlapping occurrences of the SPOP $122'1'$ is given by

$$D(x,y)=\frac{\frac{1}{2}+\frac{1}{4}\tan x(1+e^{2x}+2e^x\sin x)+\frac{1}{2}e^x\cos x}{1-y(1+(x-1)(\frac{1}{2}+\frac{1}{4}\tan x(1+e^{2x}+2e^x\sin x)+\frac{1}{2}e^x\cos x))}.$$

4.5 q-analogues for non-overlapping SPOPs

We fix some notations. Let p be a segmented POP (SPOP) and $A^p_{n,k}$ be the number of n-permutations avoiding p and having k inversions. As usually, $[n]_q=q^0+\cdots+q^{n-1}$, $[n]_q!=[n]_q\cdots[1]_q$, $\begin{bmatrix} n \\ i \end{bmatrix}_q=\frac{[n]_q!}{[i]_q![n-i]_q!}$, and, as above, $\mathrm{inv}(\pi)$ denotes the number of inversions in a permutation π. We set $A^p_n(q)=\sum_{\pi \text{ avoids } p} q^{\mathrm{inv}(\pi)}$. Moreover,

$$A^p_q(x)=\sum_{n,k}A^p_{n,k}q^k\frac{x^n}{[n]_q!}=\sum_n A^p_n(q)\frac{x^n}{[n]_q!}=\sum_{\pi \text{ avoids } p} q^{\mathrm{inv}(\pi)}\frac{x^{|\pi|}}{[|\pi|]_q!}.$$

All the definitions above are similar in case of permutations that *quasi-avoid* p, indicated by B rather than A, namely, those permutations that have exactly one occurrence of p and this occurrence consists of the $|p|$ rightmost letters in the permutations.

Theorem 34. ([19, Thm. 28]; a q-analogue of [17, Thm. 28] that is valid for POPs) Let $p = p_1\text{-}\cdots\text{-}p_k$ be a multi-pattern (p_is are SPOPs, and letters of p_i and p_j are incomparable for $i \neq j$). Then

$$A_q^p(x) = \sum_{i=1}^{k} A_q^{p_i}(x) \prod_{j=1}^{i-1} B_q^{p_j}(x) = \sum_{i=1}^{k} A_q^{p_i}(x) \prod_{j=1}^{i-1} ((x-1)A_q^{p_j}(x) + 1).$$

Theorem 35. ([19, Thm. 28]; a q-analogue of [18, Thm. 16]) If $N_p(\pi)$ denotes the maximum number of non-overlapping occurrences of a SPOP p in π, then

$$\sum_{\pi} y^{N_p(\pi)} q^{inv(\pi)} \frac{x^{|\pi|}}{|\pi|!} = \frac{A_q^p(x)}{1 - yB_q^p(x)} = \frac{A_q^p(x)}{1 - y((x-1)A_q^p(x) + 1)}.$$

5 POPs in compositions

Compositions are objects closely related to words, and some of the results on POPs in compositions can be viewed as generalizations of certain results on words. In this subsection we review some of the results in [13] and [22].

5.1 Avoiding POPs in compositions

Let $\mathbb{N}$ be the set of all positive integers, and let A be any ordered finite set of positive integers, say $A = \{a_1, a_2, \ldots, a_k\}$, where $a_1 < a_2 < a_3 < \cdots < a_k$. A composition $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of *summands*, or *parts*, namely m, is called the number of *parts* of the composition. For any ordered set $A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{N}$, we denote the set of all compositions of n with parts in A (resp. with m parts in A) by C_n^A (resp. $C_{n;m}^A$). Occurrences of patterns, in particular, POPs in compositions are defined similarly to that in permutations and words.

Theorem 36. [13, Thm. 3.3] Let $A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{N}$.

1. Let ϕ be a *shuffle pattern* τ-ℓ-ν, that is, ℓ is the largest element in the pattern while each letter in τ is incomparable to any letter in ν. Then for all $k \geq \ell$,

$$C_\phi^A(x,y) = \frac{C_\phi^{A-\{a_k\}}(x,y) - x^{a_k} y C_\tau^{A-\{a_k\}}(x,y) C_\nu^{A-\{a_k\}}(x,y)}{(1 - x^{a_k} y C_\tau^{A-\{a_k\}}(x,y))(1 - x^{a_k} y C_\nu^{A-\{a_k\}}(x,y))}.$$

2. Let ψ be a POP τ-1-ν, where 1 is the smallest element in the pattern while each letter in τ is incomparable to any letter in ν. Then for all $k \geq \ell$,

$$C_{\psi}^{A}(x,y) = \frac{C_{\psi}^{A-\{a_1\}}(x,y) - x^{a_1} y C_{\tau}^{A-\{a_1\}}(x,y) C_{\nu}^{A-\{a_1\}}(x,y)}{(1 - x^{a_1} y C_{\tau}^{A-\{a_1\}}(x,y))(1 - x^{a_1} y C_{\nu}^{A-\{a_1\}}(x,y))}.$$

Theorem 37. [13, Thm. 3.7] *Let $A \subseteq \mathbb{N}$ and let $\tau = \tau_1$-τ_2-$\cdots$-τ_s be a multi-pattern (see Subsection 3.4). Then*

$$C_{\tau}^{A}(x,y) = \sum_{j=1}^{s} C_{\tau_j}^{A}(x,y) \prod_{i=1}^{j-1} \left[\left(y \sum_{a \in A} x^a - 1 \right) C_{\tau_i}^{A}(x,y) + 1 \right].$$

Theorem 38. [13, Thm. 4.1] *Let A be any ordered set of positive integers and let τ be a consecutive pattern. Then*

$$\sum_{n,m \geq 0} \sum_{\sigma \in C_{n;m}^{A}} t^{N_\tau(\sigma)} x^n y^m = \frac{C_{\tau}^{A}(x,y)}{1 - t\left[\left(y \sum_{a \in A} x^a - 1 \right) C_{\tau}^{A}(x,y) + 1 \right]},$$

where $N_\tau(\sigma)$ is the maximum number of non-overlapping occurrences of τ in σ.

5.2 Counting POPs in compositions

While dealing with counting patterns in some objects, say, permutations, we typically solve the following problem: "find the number of permutations containing certain number of occurrences of a given pattern." In [22] another problem related to counting patterns was considered: "given a POP, how many times it occurs among all compositions?" Such studies generalize some of results in the literature, for example, those in [14] (see [22, Introd.]). To state results in this direction, we need some definitions.

Given a SPOP $w = w_1 w_2 \cdots w_m$ with m parts, let $c_w(n, \ell, s)$ be the number of occurrences of w among compositions of n with $\ell + m$ parts such that the sum of the parts preceding the occurrence is s. Let $\Omega_w(x,y,z)$ be the generating function for $c_w(n,\ell,s)$:

$$\Omega_w(x,y,z) = \sum_{n,\ell,s \in \mathbb{N}} c_w(n,\ell,s) x^n y^\ell z^s.$$

Given a segmented pattern v and $n \in \mathbb{N}$, let $P_v(n)$ denote the number of compositions of n that are order isomorphic to v. If j is the largest letter of v, then $P_v(n)$ is the number of integral solutions $t_1, \ldots, t_j$ to the system

$$\mu_1 t_1 + \cdots + \mu_j t_j = n, \qquad 0 < t_1 < \cdots < t_j, \tag{3}$$

where μ_k is the number of k's in v. We call $\mu = (\mu_1, \ldots, \mu_j)$ the *content vector* of v. By expanding terms into geometric series, one can see that the number of integral solutions to (3) is the coefficient of x^n in

$$\mathcal{P}_v(x) = \prod_{k=1}^{j} \frac{x^{m_k}}{1 - x^{m_k}}, \tag{4}$$

where $m_k = \mu_{j-k+1} + \cdots + \mu_j$ for $1 \leq k \leq j$.

Theorem 39. *Let w be a SPOP. Then*

$$\Omega_w(x, y, z) = \frac{(1-x)(1-xz)}{(1-x-xy)(1-xz-xyz)} \sum_v \mathcal{P}_v(x) \tag{5}$$

where the sum is over all linear extensions v of w.

6 Concluding remarks

The study of POPs, being a natural generalization of considering generalized patterns in permutations and words, is not only dealing with challenging enumerative problems, but also with ways to discover new connections between restricted permutations/words/compositions and other combinatorial objects. There are infinitely many partially ordered sets and patterns, which provides many opportunities for further research on POPs. Some open problems on POPs can be found in [19, Sec. 5]. We expect that POPs will play a major role in research on (permutation) patterns in the future.

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