Alternation Graphs

Magnús M. Halldórsson[1], Sergey Kitaev[12], and Artem Pyatkin[3]

[1] School of Computer Science, Reykjavik University, 101 Reykjavik, Iceland. e-mail: {mmh,sergey}@ru.is
[2] Department of Computer and Information Sciences, University of Strathclyde, Glasgow, G1 1XH, UK.
[3] School of Engineering and Computing Sciences, Durham University, Science Laboratories, South Road, Durham DH1 3LE, UK. artempyatkin@gmail.com

Abstract. A graph $G = (V, E)$ is an *alternation graph* if there exists a word W over the alphabet V such that letters x and y alternate in W if and only if $(x, y) \in E$ for each $x \neq y$.
In this paper we give an effective characterization of alternation graphs in terms of orientations. Namely, we show that a graph is an alternation graph if and only if it admits a *semi-transitive orientation* defined in the paper. This allows us to prove a number of results about alternation graphs, in particular showing that the recognition problem is in NP, and that alternation graphs include all 3-colorable graphs.
We also explore bounds on the size of the word representation of the graph. A graph G is a k-*alternation* graph if it is represented by a word in which each letter occurs exactly k times; the alternation number of G is the minimum k for which G is a k-alternation graph. We show that the alternation number is always at most n, while there exist graphs for which it is $n/2$.

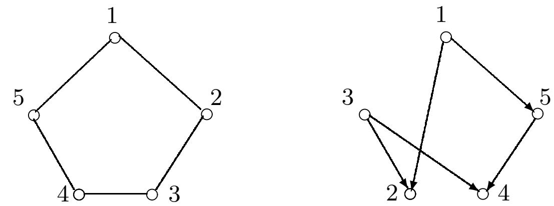
1 Introduction

Consider a scenario with n recurring tasks with requirements on the alternation of certain pairs of tasks. This captures typical situations in periodic scheduling, where there are recurring *precedence* requirements.

When tasks occur only once, the pairwise requirements form precedence constraints, which are modeled by partial orders. When the orientation of the constraints is omitted, the resulting pairwise constraints form comparability graphs. The focus of this paper is to study the class of undirected graphs induced by the alternation relationship of recurring tasks.

Consider, e.g., the following five tasks that may be involved in the operation of a given machine: 1) Initialize controller, 2) Drain excess fluid, 3) Obtain permission from supervisor, 4) Ignite motor, 5) Check oil level. Tasks 1 & 2, 2 & 3, 3 &4, 4 & 5, and 5 & 1 are expected to alternate between all repetitions of the events. This is shown in Fig. 1(b). One possible task execution sequence that obeys these recurrence constraints – and no other – is shown in Fig. 1(a). We introduce later an orientation of such graphs that will be called semi-transitive.

3 1 2 5 1 4 3 5 4 2



(a) Task execution sequence (b) An alternation graph (c) Semi-transitive orientation

Fig. 1. The word in (a) corresponds to the alternation graph in (b). A semi-transitive orientation of the graph is given in (c).

Execution sequences of recurring tasks can be viewed as words over an alphabet V, where V is the set of tasks. A graph $G = (V, E)$ is an *alternation graph* if there exists a word W over the alphabet V such that letters x and y alternate in W if and only if $(x, y) \in E$ for each $x \neq y$. If each letter appears exactly k times in the word, the graph is said to be a *k-alternation graph.* It is known that any alternation graph is a k-alternation graph for some k [10]. Alternation graphs are also known as representable graphs [11, 10, 5].

Our results. We introduce the following notion. A directed graph (digraph) $G = (V, E)$ is *semi-transitive* if it is acyclic and for any directed path $v_1 v_2 ... v_k$ either $v_1 v_k \notin E$ or $v_i v_j \in E$ for all $1 \leq i < j \leq k$. Clearly, all transitive (i.e., comparability) graphs are semi-transitive.

The main result of this paper is that the graph is an alternation graph if and only if it admits a semi-transitive orientation. This result allows us to make progress on the three most fundamental issues about alternation graphs:

– Which types of graphs are alternation graphs and which ones are not?
– How large words can be needed to represent alternation graphs?
– Are there alternative representations of these graphs that aid in reasoning about their properties?

We show that the class of alternation graphs captures non-trivial graph properties. In particular, all 3-colorable graphs are alternation graphs, whereas various types of 4-chromatic graphs cannot all be represented in this way. This resolves a conjecture of [10] regarding the Petersen graph, showing that it is an alternation graph. The result also properly captures all the previously known classes of alternation graphs: outerplanar, prisms, and comparability graphs.

Finally, we show that any alternation graph on n vertices is an n-alternation graph, again utilizing the semi-transitive orientability. This result implies that the problem of deciding whether a given graph is an alternation graph is contained in NP. Previously, no polynomial upper bound was known on the alternation number, which is the smallest value k such that the given graph is k-alternation. This bound on the alternation number is tight up to a constant factor, as we construct graphs with alternation number $n/2$. We also show that deciding if an alternation graph is k-alternation is NP-complete for $3 \leq k \leq n/2$, while the polynomially decidable class of circle graphs coincides with the class of graphs with alternation number at most 2.

Related work. Several graph classes are defined in terms of interrelationships between letters in words, where the vertices represent the letters. *Circle graphs* are those whose vertices can be represented as chords on a circle in such a way that two nodes in the graph are adjacent if and only if the corresponding chords overlap. By viewing each chord as a letter and listing the chords in order of appearance on the circle we find that these graphs correspond to words where each letter appears twice and two nodes are adjacent if and only if the letter occurrences alternate [2]. They therefore correspond to 2-alternation graphs in our vocabulary.

This has been generalized to polygon-circle graphs (see [14]), which are the intersection graphs of polygons inscribed in a circle. If we view each polygon as a letter and read the incidences of the polygons on the circle in order, we see that two polygons intersect if and only if there *exists* a pair of occurrences of the two polygons that alternate. This compares with alternation graphs where *all* occurrences of the two letters must alternate in order for the nodes to be adjacent.

The notion of directed alternation graphs was introduced in [11] to obtain asymptotic bounds on the free spectrum of the widely-studied Perkins semigroup which has played central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. The class of alternation graphs is known to contain comparability graphs [11]; in fact, the comparability graphs are precisely the permutational alternation graphs (see Sec. 2). In [10] numerous properties of alternation graphs were derived and several types of alternation and non-alternation graphs pinpointed. In particular, outerplanar graphs, prisms and 3-subdivision graphs are all alternation graphs. Also, the neighborhood of each vertex in an alternation graph induces a comparability graph. Some open questions from [10] were resolved recently in [5], including the representability of the Petersen graph. These works however do not give alternative representations or essential structural characteristics of alternation graphs.

Cyclic (or periodic) scheduling problems have been studied extensively in the operations research literature [6, 7, 13], as well as in the AI literature [3]. These are typically formulated with more general constraints, where, e.g., the 10th occurrence of task A must be preceded by the 5th occurrence of task B. The focus of this work is then on obtaining effective periodic schedules, while maintaining a small cycle time. We are, however, not aware of work on characterizing the graphs formed by the cyclic precedence constraints.

A different periodic scheduling application related to alternation graphs was considered by Graham and Zang [4], involving a counting problem related to the cyclic movements of a robot arm. More generally, given a set of jobs to be performed periodically, certain pairs (a, b) must be done alternately, e.g. since the product of job a is used as a resource for job b. Any valid execution sequence corresponds to a word over the alphabet formed by the jobs. The alternation graph given by the word must then contain the constraint pairs as a subgraph.

Organization. The paper is organized as follows. In Section 2 we give definitions of objects of interest and review some of the known results. In Section 3 we give a characterization of alternation graphs in terms of orientations and discuss some important corollaries of this fact. In Section 4 we examine the alternation number, and show that it is always at most n but can be as much as $n/2$. We explore in Section 5 which classes of graphs are alternation graphs, showing, in particular, that 3-colorable graphs are alternation graphs, but numerous other properties are orthogonal to the alternation property. The construction for triangle-free non-alternation graphs is also presented there. Finally, we conclude with a discussion of algorithmic complexity and some open problems in Section 6.

2 Definitions, notation, and known results

In this section we follow [10] to define the objects of interest.

Let W be a finite word. If W involves the letters $x_1, x_2, \dots, x_n$ then we write $Var(W) = \{x_1, \dots, x_n\}$. A word is *k-uniform* if each letter appears in it exactly k times. A 1-uniform word is also called a *permutation*. Denote by W_1W_2 the concatenation of words W_1 and W_2. We say that the letters x_i and x_j *alternate* in W if the word induced by these two letters contains neither x_ix_i nor x_jx_j as a factor. If a word W contains k copies of a letter x then we denote these k appearances of x by $x^1, x^2, \dots, x^k$. We write $x_i^j < x_k^l$ if x_i^j occurs in W before x_k^l, i. e., x_i^j is to the left of x_k^l in W.

We say that a word W *represents* the graph $G = (V, E)$ if there is a bijection $\phi : Var(W) \to V$ such that $(\phi(x_i), \phi(x_j)) \in E$ if and only if x_i and x_j alternate in W. We call a graph G an *alternation graph* if there exists a word W that represents G. It is convenient to identify the vertices of an alternation graph and the corresponding letters of a word representing it. If G can be represented by a k-uniform word, then we say that G is a *k-alternation* graph. The *alternation number* of an alternation graph G is the minimum k such that G is a k-alternation graph. We call a graph a *permutational alternation* graph if it can be represented by a word of the form $P_1P_2 \dots P_k$ where all P_i are permutations.

A digraph is *transitive* if the adjacency relation is transitive, i. e. for every vertices $x, y, z \in V$, the existence of the arcs $xy, yz \in E$ yields that $xz \in E$. A *comparability graph* is an undirected graph having an orientation of the edges that yields a transitive digraph.

The following properties of alternation graphs are useful [10]. A graph G is an alternation graph if and only if it is k-alternation for some k. If $W = AB$ is k-uniform word representing a graph G, then the word $W' = BA$ also k-represents G.

The wheel W_5 is the smallest non-alternation graph. The non-alternation graphs on 6 and 7 vertices (from [10]) are given in Fig. 2.

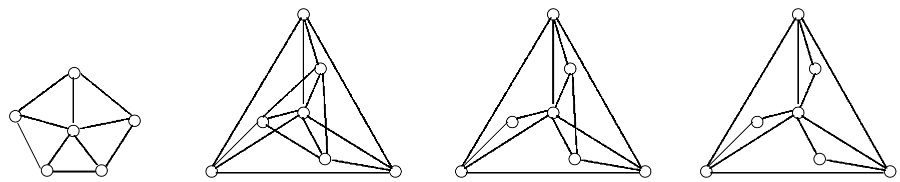


Fig. 2. Small non-alternation graphs.

3 Characterization of Alternation Graphs by Orientability

The word representation of alternation graphs is simple and natural. Yet it does not lend itself to easy arguments for the characteristic of alternation graphs. Non-alternation is even harder to argue. The main result of this section is a new characterization of alternation graphs that is effective algorithmically.

We give a characterization in terms of orientability, which implies that alternation corresponds to a property of a digraph obtained by directing the edges in certain way. It is known that a graph is a permutational alternation graph if and only if it has a transitive orientation (i.e., is a comparability graph) [11]. We prove a similar fact on alternation graphs, namely, that a graph is an alternation graph if and only if it has a certain *semi-transitive orientation* that we shall define. Our definition, in fact, generalizes that of a transitive orientation.

Other orientations have been defined in order to capture generalizations of comparability graphs. As transitive orientations form constraints on the orderings of induced P_3, these generalizations form constraints on the orderings of induced P_4. These include *perfectly orderable graphs* (and its subclasses) and *opposition graphs* [1]. Classes such as *chordal graphs* are defined in terms of vertex-orderings, and imply therefore indirectly acyclic orientations. None of these properties captures our definition below, nor does our characterization subsume any of them.

We turn to the characterization and start with definitions of certain directed graphs. A *semi-cycle* is the directed acyclic graph obtained by reversing the direction of one arc of a directed cycle. An acyclic digraph is a *shortcut* if it is induced by the vertices of a semi-cycle and contains a pair of non-adjacent vertices. Thus, a digraph on the vertex set $\{v_0, v_1, \ldots, v_t\}$, is a shortcut if it contains a directed path $v_0 v_1 \ldots v_t$, the arc $v_0 v_t$ and it is missing an arc $v_i v_j$, $0 \leq i < j \leq t$ (in particular, $t \geq 3$).

A digraph is *semi-transitive* if it is acyclic and contains no shortcuts. A graph is *semi-transitively orientable* if there exists an orientation of the edges that results in a semi-transitive graph. Our main result in this paper is the following.

Theorem 1. *A graph is an alternation graph if and only if it is semi-transitively orientable.*

We first need some additional definitions and lemmas. A *topological order* (or *topsort*) of an acyclic digraph is a permutation of the vertices that obeys the arcs, i. e. for each arc uv, u precedes v in the permutation. For a node-labeled digraph, let the topsort also refer to the word obtained by visiting the nodes in that order. Let $D = (V, E)$ be a digraph. The *t-string* digraph D^t of D is defined as follows. The vertices of D^t are v^i, for $v \in V$ and $i = 1, 2, \ldots, t$, and $v^i u^j$ is an arc in D^t if and only if either $i = j$ and $vu \in E$ or $i < j$ and $uv \in E$. Intuitively, the t-string digraph of D has t copies of D strung together. Given a word S, let G_S denote the graph represented by S. If S is a topsort of D^t then we also denote by G_S the graph represented by the word S' obtained from S by omitting the superindices of the vertices (i. e. the copies of the same vertex in S are considered as the same letters in S').

Given a digraph D, let G_D be the graph obtained by ignoring orientation.

We argue that the word representing a semi-transitive digraph comes from a special topological ordering of the t-string digraph D^t for some t. We first observe that any topological ordering of D^t preserves arcs.

Lemma 1. *Let D be a digraph with distinct node-labels. Let S be a topological ordering of a D^t. Then G_D is a subgraph of G_S.*

Proof. Consider an edge uv in G_D, and suppose without loss of generality that it is directed as uv in D. Then, in D^t, there is a directed path $u^1 v^1 u^2 v^2 \ldots u^t v^t$. Thus, occurrences of u and v in a topsort of D^t are alternating. Hence, $uv \in G_S$.

To prove equivalence, we now give a method to produce a topological ordering of D^t that generates all non-arcs. We say that an induced subgraph H *covers* a set A of non-arcs if each non-arc in A is also a non-arc in H. A word covers the non-arc if the digraph that it represents covers them.

Lemma 2. *The non-arcs incident with a path in a semi-transitive digraph can be covered with a 2-uniform word.*

Proof. Let P be a path in a semi-transitive digraph D. We shall form a topsort of the 2-string digraph D^2 and show that it covers all non-arcs having at least one endpoint on P. Let P^1 (P^2) be the first (second) copy of P in D^2. Observe that any topsort of D^2 must list the nodes in P^1 before the nodes in P^2, and each copy in order.

We say that a node x of D^2 *depends on* node y, and denote it by $y \rightsquigarrow x$, if there is a directed path from y to x in D^2, i. e. y must appear before x in a topological ordering of D^2.

Let S be any topological ordering of D^2 satisfying the following two constraints on pairs x, y of nodes in D^2:

1. if $x^1 \in P^1$ and y is listed after x^1, then y depends on x^1.
2. if $x^2 \in P^2$ and y is listed before x^2 but after x^1 (the corresponding node in P^1), then x^2 depends on y.

The ordering of other nodes is arbitrary, within these constraints. Intuitively speaking, the nodes in P^1 are listed as late as possible, while the nodes in P^2 are listed as early as possible.

We claim that this word S covers all non-arcs involving nodes in P. Consider a pair u, v, where $uv \notin G_D$ and $u \in P$. Note that v may also belong to P, in which case we may assume that the path goes from u to v. Observe that u may depend on v, or vice versa, but not both. Let u^1, v^1, u^2, v^2 be the corresponding vertices of D^2. There are three cases to consider.

Case (i): There is a path from u to v in D. We claim that u^2 does not depend on v^1. Suppose it does, i. e. $v^1 \rightsquigarrow u^2$. Then, there is an arc $x^1y^2 \in D^2$ such that $v^1 \rightsquigarrow x^1$ and $y^2 \rightsquigarrow u^2$. By the assumptions and the symmetry of the two copies of D in D^2, it follows that $y^1 \rightsquigarrow u^1 \rightsquigarrow v^1 \rightsquigarrow x^1$. By the definition of 2-string graphs, yx is an arc in D, so $y^1x^1 \in E(D^2)$. Then, by semi-transitivity, $u^1v^1 \in E(D^2)$, which implies that $uv \in E(G_D)$, which is a contradiction. It now follows that the nodes will occur as $u^1u^2v^1v^2$ in S, i. e. $uv \notin E(G_S)$.

Case (ii): There is a path from v to u in D. This is symmetric to case (i), with u replaced by v. Thus, the nodes will occur as $v^1v^2u^1u^2$ in S.

Case (iii): The nodes u and v are incomparable in D. In particular, v is not in P. Then, u^1 and v^1 do not depend on each other, nor do u^2 and v^2. If v^2 depends on u^1 then the nodes occur as $v^1u^1u^2v^2$ in S. Otherwise, their order is $v^1v^2u^1u^2$.

We now return to the proof of Theorem 1, starting with the forward direction. Given a word S, we direct an edge of G_S from x to y if the first occurrence of x is before that of y in the word. Let us show that such an orientation D of G_S is semi-transitive. Indeed, assume that $x_0x_t \in E(D)$ and there is a directed path $x_0x_1 \ldots x_t$ in D. Then in the word S we have $x_0^i < x_1^i < \ldots < x_t^i$ for every i. Since $x_0x_t \in E(D)$ we have $x_t^i < x_0^{i+1}$. But then for every $j < k$ and i there must be $x_j^i < x_k^i < x_j^{i+1}$, i. e. $x_ix_j \in E(D)$. So, D is semi-transitive.

For the other direction, denote by G the graph and by D its semi-transitive orientation. Let $P_1, P_2, \ldots, P_\tau$ be the set of directed paths covering all vertices of D. For every $i = 1, 2, \ldots, \tau$ denote by S_i the topsort of the digraph D^2 satisfying the conditions of Lemma 2 for the path P_i. Put $S = S_1S_2 \ldots S_\tau$. Clearly, S is a 2τ-uniform word; it can be treated as a topsort of a 2τ-string $D^{2\tau}$. Then $G = G_S$. Indeed, by Lemma 1 we have $E(G) \subset E(G_S)$. On the other hand, if

$uv \notin E(G)$ then $u \in P_i$ for some i, and thus by Lemma 2 the letters u and v are not alternating in the subword S_i. Therefore, $uv \notin E(S)$. Theorem 1 is proved. □

Theorem 1 makes clear the relationship to comparability graphs, which are those that have transitive orientations. Since transitive digraphs are also semi-transitive, this immediately implies that comparability graphs are alternation graphs.

The construction in Lemma 2 shows that all alternation graphs can be represented "almost" permutationally. This is made more precise as follows.

Observation 2 *Let G be an alternation graph. Then there is a word W representing G such that for any prefix P of W and any pair a, b of letters, the number of occurrences of a and b in P differ by at most two.*

4 The Alternation Number of Graphs

We focus now on the following question: Given an alternation graph, how large is its alternation number? In [10], certain classes of graphs were proved to be 2- or 3-alternation, and an example was given of a graph (the triangular prism) with the alternation number of 3. On the other hand, no examples were known of graphs with alternation numbers larger than 3, nor were there any non-trivial upper bounds known. We show here that the maximum alternation number of alternation graphs is linear in the number of vertices.

For the upper bound, we use the results of the preceding section. We have the following directly from the proof of Theorem 1.

Corollary 1. *An alternation graph G is a $2\tau(G)$-alternation graph, where $\tau(G)$ is the minimum number of paths covering all nodes in some semi-transitive orientation of G.*

This immediately gives an upper bound of $2n$ on the alternation number. We can improve this somewhat with an effective procedure.

Theorem 3. *Given a semi-transitive digraph D on n vertices, there is a polynomial time algorithm that generates an n-uniform word representing G_D. Thus, each alternation graph is an n-alternation graph.*

Proof. The algorithm works as follows.

Step 0. Start with $A = \emptyset$ and $i = 1$.

Step i. If D contains a path P_i covering at least two vertices from $V \setminus A$ then let $A := A \cup V(P_i)$ and $i := i + 1$. Otherwise, let $B = V \setminus A$ and go to the Final Step.

Final Step. Let S_i be the topsort of the digraph D^2 satisfying the conditions of Lemma 2 for the path P_i and put $S' = S_1 S_2 \ldots S_t$ where t is the number of paths found at previous steps. If $|B| \leq 1$ then let $S = S'$. Otherwise, consider a topsort S_0 of D where the vertices of B are listed in a row (since the vertices of

B do not depend on each other, such a topsort must exist) and in particular in the reverse order of their appearance in S_1. Let $S = S'S_0$.

Clearly, $G_D = G_S$ (the proof is the same as in Theorem 1). It is easy to verify that each letter appears in S at most n times.

Theorem 3 implies that the graph property of alternation is polynomially verifiable, answering an open question in [10]. Indeed, having an alternation graph G, we may ask for a word representing it and verify this fact in time bounded by the polynomial in n.

Corollary 2. *The recognition problem for alternation graphs is in NP.*

We now show that there are graphs with alternation number of $n/2$, matching the upper bound within a factor of 2.

The *crown graph* $H_{k,k}$ is the graph obtained from the complete bipartite graph $K_{k,k}$ by removing a perfect matching. Denote by G_k the graph obtained from a crown graph $H_{k,k}$ by adding an all-adjacent vertex.

Theorem 4. *The graph G_k has alternation number $k = \lfloor n/2 \rfloor$.*

The proof is based on three statements; the proof of the first is given in the appendix.

Lemma 3. *Let H be a graph and G be the graph obtained from H by adding an all-adjacent vertex. Then G is a k-alternation graph if and only if H is a permutational k-alternation graph.*

Lemma 4. *A comparability graph is permutational k-alternation graph if and only if the poset induced by this graph has dimension at most k.*

Proof. Let H be a comparability graph and W be a word permutationally k-representing it. Each permutation in W can be considered as a linear order where $a < b$ if a meets before b in the permutation (and vice versa). We want to show that the comparability graph of the poset induced by the intersection of these linear orders coincides with H.

Two vertices a and b are adjacent in H if and only if their letters alternate in the word. So, they must be in the same order in each permutation, i. e. either $a < b$ in every linear order or $b < a$ in every linear order. But this means that a and b are comparable in the poset induced by the intersection of the linear orders, i. e. a and b are adjacent in its comparability graph.

Lemma 5 ([9]). *The poset P over $2k$ elements $\{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k\}$ such that $a_i < b_j$ for every $i \neq j$ and all other elements are not comparable has dimension k.*

Now we can prove Theorem 4.

Proof. Since the crown graph $H_{k,k}$ is a comparability graph of the poset P, we deduce from Lemmas 5 and 4 that $H_{k,k}$ is permutational k-alternation graph but not a permutational $(k-1)$-alternation graph. Then by Lemma 3 we have that G_k is a k-alternation graph but not a $(k-1)$-alternation graph. Theorem 4 is proved. □

The above arguments help us also in deciding the complexity of determining the alternation number. From Lemmas 3 and 4, we see that it is as hard as determining the dimension k of a poset. Yannakakis [16] showed that the latter is NP-hard, for any $3 \leq k \leq \lceil n/2 \rceil$. We therefore obtain the following.

Proposition 1. *Deciding whether a given graph is a k-alternation graph, for any given $3 \leq k \leq \lceil n/2 \rceil$, is NP-complete.*

It was further shown by Hegde and Jain [8] that it is NP-hard to approximate the dimension of a poset within almost a square root factor. We therefore obtain the same hardness for the alternation number.

Proposition 2. *Approximating the alternation number within $n^{1/2-\epsilon}$-factor is NP-hard, for any $\epsilon > 0$.*

5 Characteristics of Alternation Graphs

When faced with a new graph class, the most basic questions involve the kind of properties it satisfies: which known classes are properly contained (and which not), which graphs are otherwise contained (and which not), what operations preserve alternation (or non-alternation), and which properties hold for these graphs.

Previously, it was known that the class of alternation graphs includes comparability graphs, outerplanar graphs, subdivision graphs, and prisms. The purpose of this section is to clarify this situation significantly, including resolving some conjectures. We start with exploring the relation of colorability and alternation.

Theorem 5. *3-colorable graphs are semi-transitively orientable, and thus alternation graphs.*

Proof. Given a 3-coloring of a graph, direct its edges from the first color class through the second to the third class. It is easy to see that we obtain a semi-transitive digraph.

This implies a number of earlier results on alternation, including that of outerplanar graphs, subdivision graphs, and prisms. The theorem also shows that 2-degenerate graphs, graphs of maximum degree 3 (via Brooks theorem), and triangle-free planar graphs (via Grötzch's theorem) are all alternation graphs.

This result does not extend to higher chromatic numbers. The examples in Fig. 2 show that 4-colorable graphs can be non-alternation. We can, however, obtain a result in terms of the *girth* of the graph, which is the length of its shortest cycle.

Proposition 3. *Let G be a graph whose girth is greater than its chromatic number. Then, G is an alternation graph.*

Proof. Suppose the graph is colored with $\chi(G)$ natural numbers. Orient the edges of the graph from small to large colors. There is no directed path with more than $\chi(G) - 1$ arcs, but since G contains no cycle of $\chi(G)$ or fewer edges, there can be no shortcut. Hence, the digraph is semi-transitive.

The next theorem shows us how to construct an infinite series of triangle-free non-alternation graphs. This answers an open question in [10].

Theorem 6. *There exist triangle-free non-alternation graphs.*

Proof. Let H be a 4-chromatic graph with girth at least 10 (such graphs exist by Erdös theorem). For every path P of length 3 in H add to H the edge e_P connecting its ends. Denote the obtained graph by G. Let us show that G is a triangle-free non-alternation graph.

If G contains a triangle on the vertices u, v, w then H contains three paths P_{uv}, P_{uw}, and P_{vw} of lengths 1 or 3 connecting these vertices. Let T be a graph spanned by these three paths. Since T has at most 9 edges and the girth of H is at least 10, T is a tree. Clearly, it cannot be a path. So, it is a subdivision of $K_{1,3}$ with the leafs u, v, w. But then at least one of the paths P_{uv}, P_{uw}, P_{vw} must have an even length, a contradiction.

So, G is triangle-free. Assume that G has a semi-transitive orientation. Then it induces a semi-transitive orientation on H. Since H is 4-chromatic, each of its acyclic orientation must contain a directed path P of length at least 3. But then the orientation of the edge e_P in G produces either a 4-cycle or a shortcut, contradicting the semi-transitivity. So, G is a triangle-free non-alternation graph.

6 Concluding Remarks and Open Questions

It is natural to ask about optimization problems on alternation graphs. Theorem 5 implies that many classical optimization problems are NP-hard on alternation graphs:

Observation 7 *The optimization problems Independent Set, Dominating Set, Graph Coloring, Clique Partition, Clique Covering are NP-hard on alternation graphs.*

Note that it may be relevant whether the representation of the graph as a semi-transitive digraph is given; solvability under these conditions is open. However, some problems remain polynomially solvable:

Observation 8 *The Clique problem is polynomially solvable on alternation graphs.*

Indeed, we can simply use the fact that the neighborhood of any node is a comparability graph. The clique problem is easily solvable on comparability graphs. Thus, it suffices to search for the largest clique within all induced neighborhoods.

We conclude with several open questions about alternation graphs:

1. Is it NP-hard to decide whether a graph is an alternation graph?
2. What is the maximum alternation number of a graph? We know that it lies between $n/2$ and n.
3. Are all graphs of maximum degree 4 alternation graphs?
4. Is there an algorithm that forms an $f(k)$-representation of a k-alternation graph, for some function f? Namely, can the alternation number be approximated as a function of itself? The same question holds also for the partial order (or poset) dimension [8].

References

1. A. Brandstädt, V. Bang Lee, J. P. Spinrad. *Graph Classes: A Survey.* Monographs on Discrete Mathematics and Applications. SIAM, 1987.
2. B. Courcelle. Circle graphs and Monadic Second-order logic. *J. Applied Logic* 6(3): 416-442, 2008.
3. D. L. Draper, A. K. Jonsson, D. P. Clements and D. E. Joslin. Cyclic Scheduling. In *Proc. IJCAI*, 1999.
4. R. Graham and N. Zang, Enumerating split-pair arrangements, *J. Combin. Theory, Series A* **115**, Issue 2 (2008), 293–303.
5. M. Halldórsson, S. Kitaev and A. Pyatkin. Graphs capturing alternations in words. In *Proc. 14th Developments in Language Theory (DLT)*, LNCS 6224, pages 436–437, August 2010.
6. C. Hanen, A. M. Kordon. Periodic schedules for linear precedence constraints. Discrete Applied Mathematics 157(2):280–291, 2009.
7. C. Hanen and A. Munier. Cyclic scheduling on parallel processors: An overview. In *P. Chretienne, E. G. Coffman, Jr., J. K. Lenstra, and Z. Liu, editors, Scheduling Theory and its Applications*, chapter 9. John Wiley & Sons, 1995.
8. R. Hegde and K. Jain. The hardness of approximating poset dimension. *Electronic Notes in Discrete Mathematics* **29** (2007), 435–443.
9. *Order dimension*. In Wikipedia: The Free Encyclopedia `http://en.wikipedia.org/wiki/Order_dimension`, retrieved 2 February 2011.
10. S. Kitaev, A. Pyatkin. On representable graphs. *Automata, Languages and Combinatorics* **13** (2008) 1, 45–54.
11. S. Kitaev and S. Seif. Word problem of the Perkins semigroup via directed acyclic graphs. *Order*, DOI 10.1007/s11083-008-9083-7 (2008).
12. A. Konovalov, S. Linton. Search of representable graphs with constraint solvers, in preparation.
13. M. Middendorf and V. G. Timkovsky. On scheduling cycle shops: Classification, complexity and approximation. *Journal of Scheduling*, 5:135–169, 2002.
14. M. Pergel. Recognition of polygon-circle graphs and graphs of interval filament is NP-complete. In *Proc. 33rd Intl. Workshop on Graph-Theoretic Concepts in Computer Science (WG)*, Lecture Notes in Computer Science 4769, Springer, 2007.
15. Web page on small graphs. `http://wwwteo.informatik.uni-rostock.de/isgci/smallgraphs.html`
16. M. Yannakakis. The complexity of the partial order dimension problem. *SIAM J. Algebraic Discrete Methods* 3(3) (1982), pp. 351–358.

Appendix: Proof Missing in the Main Text

Lemma 3. *Let H be a graph and G be the graph obtained from H by adding an all-adjacent vertex. Then G is a k-alternation graph if and only if H is permutationally k-alternation.*

Proof. Let 0 be the letter corresponding to the all-adjacent vertex. Then every other letter of the word W representing G must appear exactly once between two consecutive zeroes. We may assume also that W starts with 0. Then the word $W \setminus \{0\}$, formed by deleting all occurrences of 0 from W, is a permutational k-representation of H. Conversely, if W' is a word permutationally k-representing H, then we insert 0 in front of each permutation to get a (permutational) k-representation of G.