Counting ordered patterns in words generated by morphisms

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Joint work with Toufik Mansour and Patrice Séébold

Occurrences of the “classical" pattern 1-3-2 in 13524:

$$\mathbf{1\ 3\ 5\ 2\ 4}, \quad \mathbf{1\ 3\ 5\ 2\ 4}, \quad \mathbf{1\ 3\ 5\ 2\ 4}, \quad \mathbf{1\ 3\ 5\ 2\ 4}$$

Occurrences of the "classical" pattern 1-3-2 in 13524:

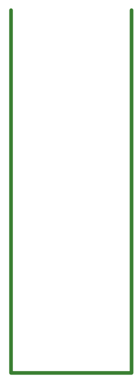
1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4

A generalized pattern is a pattern that allows the requirement that two adjacent letters in the pattern must be adjacent in the permutation.

Pattern	Occurrences in 13542
1-3-2	**1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2**
1-32	**1 3 5 4 2, 1 3 5 4 2, 1 3 5 4 2**
132	**1 3 5 4 2**

Sorting with a stack

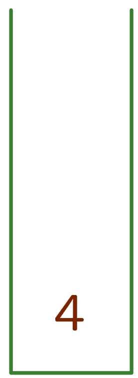
4 1 6 3 2 5



Sorting with a stack

1 6 3 2 5

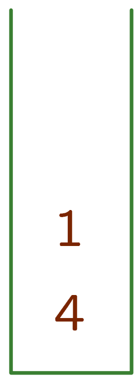
Numbers on stack
must increase
from top



Sorting with a stack

6 3 2 5

Numbers on stack
must increase
from top



Sorting with a stack

1   6 3 2 5

Numbers on stack
must increase
from top

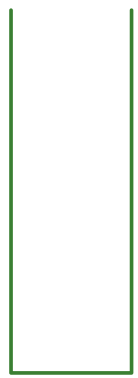
4

Sorting with a stack

1 4

6 3 2 5

Numbers on stack
must increase
from top

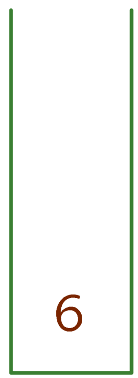


Sorting with a stack

1 4

3 2 5

Numbers on stack
must increase
from top

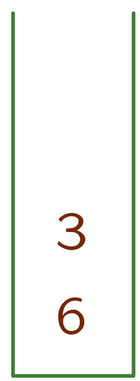


Sorting with a stack

1 4

2 5

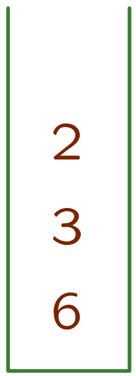
Numbers on stack
must increase
from top



Sorting with a stack

1 4 5

Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 5

Numbers on stack
must increase
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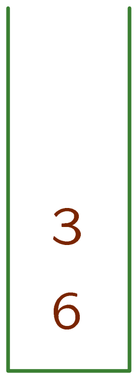
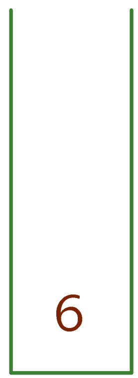


Sorting with a stack

1 4 2 3 5



Numbers on stack
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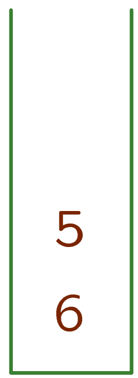
Sorting with a stack

1 4 2 3

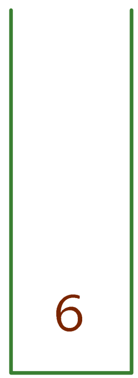
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5

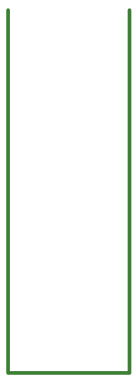
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5 6

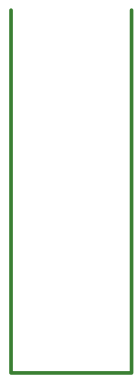
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5 6

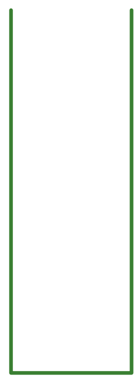
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5 6

4 1 6 3 2 5

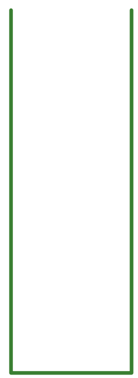


Numbers on stack
must increase
from top

Sorting with a stack

1 **4** 2 3 5 6   **4** 1 **6** **3** 2 5

Numbers on stack
must increase
from top

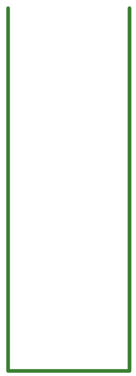


Sorting with a stack

1 4 2 3 5 6

4	1	6	3	2	5
2		3	1		

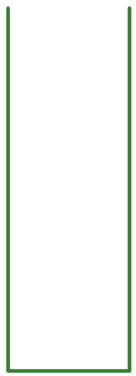
Numbers on stack
must increase
from top



Sorting with a stack

1 4 2 3 5 6

2 3 1

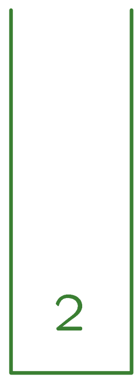


Numbers on stack
must increase
from top

Sorting with a stack

1 4 2 3 5 6

3 1



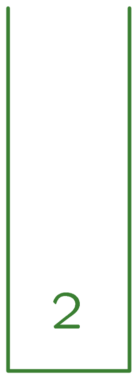
Numbers on stack
must increase
from top

Theorem. [Knuth] A permutation is stack-sortable if and only if it avoids 2-3-1.

1 4 2 3 5 6

3 1

Numbers on stack
must increase
from top



Theorem. [Knuth] A permutation is stack-sortable if and only if it avoids 2-3-1.

The number of such permutations is the n-th Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

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The number of such permutations is the n-th Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

They have the generating function

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

1969 D. Knuth: The Art of computer programming, vol. I

1985 R. Simion, F. Schmidt: Restricted permutations, European J. Combin. **6**, no. 4, 383–406.

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1985 R. Simion, F. Schmidt: Restricted permutations, European J. Combin. **6**, no. 4, 383–406.

1992 Present: Explosive growth (several hundreds papers appeared)

2002 H. Wilf: The patterns of permutations, DM **257**, 575–583.

2003 S. Kitaev, T. Mansour: Survey of certain pattern problems

2004 M. Bóna: Combinatorics of Permutations, xiv+383 pp.

Permutation Patterns:
Classical patterns: Knuth, 1969 Generalized patterns: Babson and Steingrímsson, 2000 Partially ordered patterns: SK, 2001

Word Patterns:
Classical word patterns: Burstein, 1998 Generalized word patterns: Burstein and Mansour, 2002 Partially ordered word patterns: SK and Mansour, 2003

Patterns in matrices: SK, Mansour and Vella, 2003
Patterns in n-dimensional objects: SK and Robbins, 2004

Patterns in even (odd) permutations: Simion and Schmidt, 1985
Colored patterns in colored permutations: Mansour, 2001
Signed patterns in signed permutations: Mansour and West, 2002
Patterns with respect to parity: SK and Remmel, 2005

Patterns	Related combinatorial objects	Authors
2-31	Dyck paths (Catalan numbers)	Claesson
1-23	Set partitions (Bell numbers)	Claesson
1-23, 12-3	Non-overlapping partitions (Bessel numbers)	Claesson
1-23, 1-32	Involutions	Claesson
1-23, 13-2	Motzkin paths	Claesson
132, [21	Increasing rooted trimmed trees	SK
213, 4123 3124, 2134	Matchings in the coronas of the complete graphs	SK Pyatkin
1-3-2, 21-34, 12-34, 13-24, 14-23	"Horse paths": paths with steps (0,1), (1,2), (1,1), (2,1) starting at the origin	Mansour Hou
$\sqcup$abcd, a<c, b>d (6 patterns)	First quadrand lattice walks starting at the origin with N, S, E, W steps	SK
$\sqcup 1\sigma$, σ is a perm. on $\{2,\ldots,k+1\}$	Permutations with cycles of length at most k	SK

Let Σ be an alphabet.

A map $\varphi : \Sigma^* \to \Sigma^*$ is called a morphism, if we have

$$\varphi(uv) = \varphi(u)\varphi(v)$$

for any $u, v \in \Sigma^*$.

A morphism φ can be defined by defining $\varphi(i)$ for each $i \in \Sigma$.

The Thue-Morse sequence is defined by the morphism μ:

$$\mu(0) = 01,$$
$$\mu(1) = 10.$$

0, 01, 0110, 01101001, ...

Patterns in combinatorics on words:

XX – squares, $1\underbrace{02}\underbrace{02}010$

XXX – cubes, $11\underbrace{011}\underbrace{011}\underbrace{011}0$

XYXXY – $2\underbrace{11}\underbrace{323}\underbrace{11}\underbrace{11}\underbrace{323}3$

Parikh vectors:

$\mathcal{A} = \{a_1 < a_2 < a_3\}$

$w = a_2a_1a_3a_1a_3$

The Parikh vector of w is $(|w|_{a_1}, |w|_{a_2}, |w|_{a_3}) = (2, 1, 2)$

Introduced by R.J. Parikh in 1966

Parikh matrices

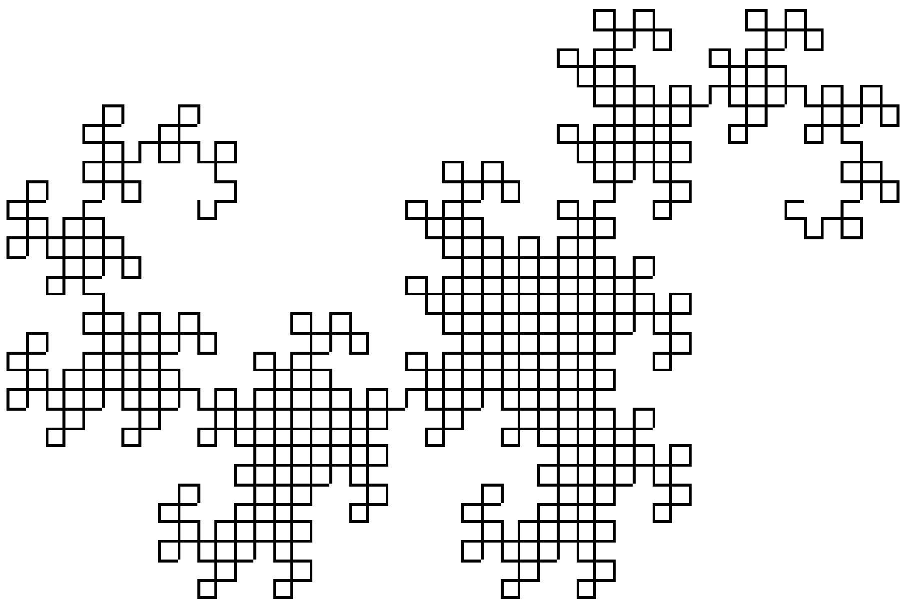
Introduced by A. Mateescu, A. Salomaa, K. Salomaa and S. Yu in 2001

$\Psi(aabbc) = \Psi(a)\Psi(a)\Psi(b)\Psi(b)\Psi(c) =$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The (Harter-Heighway) Dragon Curve (paperfolding sequence) was discovered by physicist John E. Heighway in 1967. (An example of a recursively generated fractal shape.)



$\Omega = \{A, B, a, b\}$, and γ and h are the following morphisms.

$$\begin{array}{rcl} \gamma : \Omega^* & \to & \Omega^* \\ A & \mapsto & AaB \\ B & \mapsto & AbB \\ a & \mapsto & a \\ b & \mapsto & b \end{array} \qquad \begin{array}{rcl} h : \Omega^* & \to & \{1,3\}^* \\ A & \mapsto & 1 \\ B & \mapsto & 3 \\ a & \mapsto & 1 \\ b & \mapsto & 3 \end{array}$$

Theorem. The Dragon curve w_σ is generated by the tag-system $(\Omega, A, \gamma, h, \{1,3\})$, i.e., $w_\sigma = h(\gamma^\omega(A))$.

Theorem. [SK, 2003] There does not exist a morphism whose iteration defines the Dragon curve w_σ.

Samples of counting patterns in w_σ results, [SK, 2003-2004]

Among the first $2^n - 1$ symbols of w_σ:

$\underbrace{1 - 1 - \cdots - 1}_{k}$ occurs $\frac{2^n - k}{2^{n-1} - k}\binom{2^{n-1}-1}{k}$ times

1-2 occurs $2 \cdot 4^{n-2} + (n-2) \cdot 2^{n-2}$ times

221 occurs $3 \cdot 2^{n-4} - 1$ times

12-21 occurs $\frac{1}{2}4^{n-2} - 3 \cdot 2^{n-4}$ times

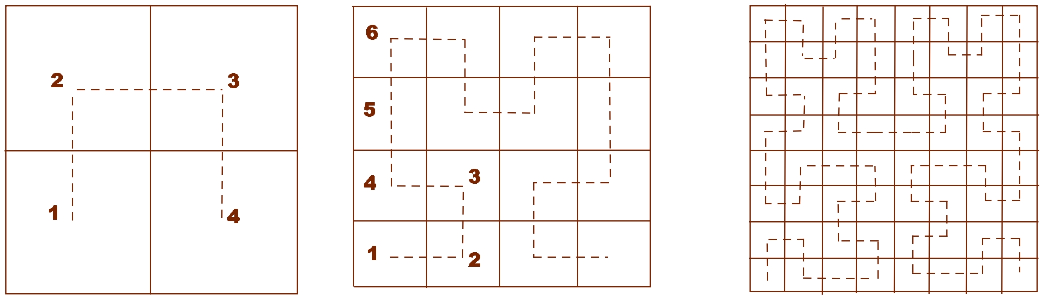
1-221 occurs $\frac{1}{2}4^{n-2} + 27 \cdot 2^{n-5} - n - 7$ times

The number c_n^τ of occurrences of 2-1-221 among the first $2^n - 1$ symbols of w_σ can be calculated using

$$\begin{pmatrix} c_n^\tau \\ d_n^\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_{n-1}^\tau \\ d_{n-1}^\tau \end{pmatrix} + \begin{pmatrix} \frac{5}{1024}8^n + \frac{25-3n}{256}4^n - \frac{171}{64}2^n + 9 \\ \frac{5}{1024}8^n + \frac{21-3n}{256}4^n - 2^{n+1} \end{pmatrix}$$

with initial conditions $c_5^\tau = 70$ and $d_5^\tau = 74$.

The Peano (Hilbert) curve (1890–1891) is an example of fractal space filling curves.



x_1 x_2 x_3

A Peano word P_n is obtained by traveling along the Peano curve after the n-th iteration. P_n is over $\Sigma = \{u, \bar{u}, r, \bar{r}\}$ where u stands for *up*, $\bar{u}$ stands for *down*, r for *right*, and $\bar{r}$ stands for *left*.

The Peano infinite word $P = \lim_{n\to\infty} P_{2n+1}$.

$\Omega = \{A, B, C, D, a, b, c, d\}$, and γ and h are the following morphisms.

$$\begin{array}{rcl} \gamma : \Omega^* & \rightarrow & \Omega^* \\ A & \mapsto & BaAbAcD \\ B & \mapsto & AbBaBdC \\ C & \mapsto & DcCdCaB \\ D & \mapsto & CdDcDbA \\ a & \mapsto & a \\ b & \mapsto & b \\ c & \mapsto & c \\ d & \mapsto & d \end{array} \qquad \begin{array}{rcl} h : \Omega^* & \rightarrow & \Sigma^* \\ A & \mapsto & ur\bar{u} \\ B & \mapsto & ru\bar{r} \\ C & \mapsto & \bar{u}\bar{r}u \\ D & \mapsto & \bar{r}\bar{u}r \\ a & \mapsto & u \\ b & \mapsto & r \\ c & \mapsto & \bar{u} \\ d & \mapsto & \bar{r} \end{array}$$

Theorem. [SK, Mansour, Séébold, 2003] P is the infinite word generated by the tag-system $(\Omega, A, \gamma^2, h, \Sigma)$, i.e., $P = h((\gamma^2)^\omega(A))$.

Theorem. [SK, Mansour, Séébold, 2003] P cannot be generated by a D0L-system, and thus cannot be generated by a morphism.

Samples of counting patterns in P_n results, [SK, Mansour, Séébold, 2003]

$$12(P_{2k+1}) = \frac{2}{5}(4 \cdot 16^k + 1),$$
$$12(P_{2k+2}) = \frac{2}{5}(16^{k+1} - 1),$$
$$21(P_{2k+1}) = \frac{8}{5}(16^k - 1),$$
$$21(P_{2k+2}) = \frac{2}{5}(16^{k+1} - 1).$$

Counting ordered patterns in words generated by morphisms

[SK, Mansour, Séébold, 2008]

$\mathcal{A} = \{a_1 < a_2 < \cdots < a_k\}$. Let f be a morphism and $n \geqslant 0$. The incidence matrix of f^n is the $k \times k$ matrix

$$M(f^n) = (m_{n,i,j})_{1 \leqslant i,j \leqslant k}$$

where $m_{n,i,j}$ is the number of occurrences of the letter a_i in the word $f^n(a_j)$.

Remark. "#" in the paper has the same meaning as "-" here.

The vector of non-inversions 1-2$(f^n) = (|f^n(a_i)|_{1-2})_{1\leqslant i\leqslant k}$.

The vector of inversions 2-1$(f^n) = (|f^n(a_i)|_{2-1})_{1\leqslant i\leqslant k}$.

The vector of p-repetitions with gaps of a letter $R_pG(f^n) = (|f^n(a_i)|_{(1-)^p})_{1\leqslant i\leqslant k}$.

For each letter $a_\ell \in \mathcal{A}$, let p_ℓ and q_ℓ be such that $f(a_\ell) = a_{\ell_1} \dots a_{\ell_{p_\ell}}$ and $f^n(a_\ell) = a_{\ell'_1} \dots a_{\ell'_{q_\ell}}$. Then, for all $n \in \mathbb{N}$,

$$|f^{n+1}(a_\ell)|_{1-2} = \sum_{1\leqslant i<j\leqslant p_\ell} (\sum_{r=1}^{k-1}(m_{n,r,\ell_i} \cdot \sum_{s=r+1}^{k} m_{n,s,\ell_j})) + \sum_{t=1}^{k} |f^n(a_t)|_{1-2} \cdot m_{1,t,\ell} \,,$$

$$= \sum_{1\leqslant i<j\leqslant q_\ell} (\sum_{r=1}^{k-1}(m_{1,r,\ell'_i} \cdot \sum_{s=r+1}^{k} m_{1,s,\ell'_j})) + \sum_{t=1}^{k} |f(a_t)|_{1-2} \cdot m_{n,t,\ell} \,,$$

$$|f^{n+1}(a_\ell)|_{2-1} = \sum_{1\leqslant i<j\leqslant p_\ell} (\sum_{r=2}^{k}(m_{n,r,\ell_i} \cdot \sum_{s=1}^{r-1} m_{n,s,\ell_j})) + \sum_{t=1}^{k} |f^n(a_t)|_{2-1} \cdot m_{1,t,\ell} \,,$$

$$= \sum_{1\leqslant i<j\leqslant q_\ell} (\sum_{r=2}^{k}(m_{1,r,\ell'_i} \cdot \sum_{s=1}^{r-1} m_{1,s,\ell'_j})) + \sum_{t=1}^{k} |f(a_t)|_{2-1} \cdot m_{n,t,\ell} \,.$$

The following is obvious.

For each letter $a_\ell \in A$ and for all $n \in \mathbb{N}$,

$$|f^n(a_\ell)|_{(1-)^p} = \sum_{t=1}^{k} \binom{m_{n,t,\ell}}{p}.$$

The Thue-Morse morphism

$\mu(0) = 01,$
$\mu(1) = 10.$

$$M(\mu^n) = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$$

For any integer $n \geqslant 2$,

$$\text{1-2}(\mu^n) = \text{2-1}(\mu^n) = \begin{bmatrix} 2^{2n-3} & 2^{2n-3} \end{bmatrix} \text{ and}$$

$$R_p G(\mu^n) = \begin{bmatrix} 2 \cdot \binom{2^{n-1}}{p} & 2 \cdot \binom{2^{n-1}}{p} \end{bmatrix}$$

The Fibonacci morphism

$\phi(a_1) = a_1 a_2$,
$\phi(a_2) = a_1$.

It generates *Fibonacci sequence* $\varphi^\omega(a_1)$.

$$M(\varphi^n) = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$$

For every integer $n \geqslant 0$,

$$\begin{aligned} |\varphi^{n+2}(a_1)|_{2\text{-}1} &= \textstyle\sum_{p=0}^{n} F_p F_{n-p}^2 \,, \\ |\varphi^{n+2}(a_1)|_{1\text{-}2} &= |\varphi^{n+2}(a_1)|_{2\text{-}1} + F_n + \begin{cases} 1 & \text{if } n \text{ is odd,} \\ -1 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

A particular family of morphisms

f involving at least 2 letters has the following properties:

1. $\exists$ a positive integer m such that $|f(a_1)|_{a_i} = m$, $1 \leqslant i \leqslant k$,

2. $\exists$ a positive integer d such that $|f(a_2 \ldots a_k)|_{a_i} = d$, $1 \leqslant i \leqslant k$,

3. $\forall\ i, j$, $1 \leqslant i, j \leqslant k$, $|f(a_i a_j)|^{ext}_{\text{1-2}} = |f(a_j a_i)|^{ext}_{\text{1-2}}$. [For example, for Thue-Morse morphism, $|\mu(a_1 a_2)|^{ext}_{\text{1-2}} = |a_1 a_2 a_2 a_1|^{ext}_{\text{1-2}} = 1$ $= |a_2 a_1 a_1 a_2|^{ext}_{\text{1-2}} = |\mu(a_2 a_1)|^{ext}_{\text{1-2}}$]

For every positive integer n,

$$
\begin{aligned}
|f^{n+1}(a_1)|_{\text{1-2}} = \;& m(d+m)^{n-1} \sum_{i=1}^{k} |f(a_i)|_{\text{1-2}} \\
&+ \frac{[m(d+m)^{n-1}-1]m(d+m)^{n-1}}{2} \sum_{j=1}^{k} |f(a_j a_j)|^{ext}_{\text{1-2}} \\
&+ m^2(d+m)^{2n-2} \sum_{1 \leqslant i<j \leqslant k} |f(a_i a_j)|^{ext}_{\text{1-2}}
\end{aligned}
$$

$$
\begin{aligned}
|f^{n+1}(a_2 \dots a_k)|_{\text{1-2}} = \;& d(d+m)^{n-1} \sum_{i=1}^{k} |f(a_i)|_{\text{1-2}} \\
&+ \frac{[d(d+m)^{n-1}-1]d(d+m)^{n-1}}{2} \sum_{j=1}^{k} |f(a_j a_j)|^{ext}_{\text{1-2}} \\
&+ d^2(d+m)^{2n-2} \sum_{1 \leqslant i<j \leqslant k} |f(a_i a_j)|^{ext}_{\text{1-2}}
\end{aligned}
$$

The Istrail morphism, 1977

The morphism h on $\mathcal{A} = \{a_1 < a_2 < a_3\}$:

$$h(a_1) = a_1 a_2 a_3, \qquad h(a_2) = a_1 a_3, \qquad h(a_3) = a_2$$

The word $h^\omega(a_1)$ is closely related to the Thue-Morse word T. If

$$\delta : \begin{array}{ccc} a_1 & \mapsto & a_1 \\ a_2 & \mapsto & a_1 a_2 \\ a_3 & \mapsto & a_1 a_2 a_2 \end{array}$$

then $T = \delta(h^\omega(a_1))$ (Lothaire, 1983).

The Istrail morphism, 1977

For $n \geqslant 1$,

$$|h^{n+1}(a_1)|_{\text{1-2}} = |h^{n+1}(a_2a_3)|_{\text{1-2}} = 3 \cdot 2^{2n-1} + 2^n.$$

$$|h^{n+1}(a_1)|_{\text{2-1}} = |h^{n+1}(a_2a_3)|_{\text{2-1}} = 3 \cdot 2^{2n-1} - 2^n.$$

The Prouhet morphisms, 1851

(A generalization of Thue-Morse morphism)

Let $k \geqslant 2$ and $\mathcal{A} = \{a_1 < \cdots < a_k\}$. The Prouhet morphism π_k is

$$\pi_k(a_i) = a_i a_{i+1} \ldots a_k a_1 \ldots a_{i-1}, \qquad 1 \leqslant i \leqslant k.$$

Let $k = 6$. The morphism π_6 is given by

$$\begin{array}{lcl} a_1 & \mapsto & a_1a_2a_3a_4a_5a_6 \\ a_2 & \mapsto & a_2a_3a_4a_5a_6a_1 \\ a_3 & \mapsto & a_3a_4a_5a_6a_1a_2 \\ a_4 & \mapsto & a_4a_5a_6a_1a_2a_3 \\ a_5 & \mapsto & a_5a_6a_1a_2a_3a_4 \\ a_6 & \mapsto & a_6a_1a_2a_3a_4a_5 \end{array}$$

The Prouhet morphisms, 1851

For every i, $1 \leqslant i \leqslant k$, and for every positive integer n,

$$|\pi_k^{n+1}(a_i)|_{\text{1-2}} = \frac{(k-1)k^n}{12}\left(3k^{n+1} + k - 2\right),$$

$$|\pi_k^{n+1}(a_i)|_{\text{2-1}} = \frac{(k-1)k^n}{12}\left(3k^{n+1} - k + 2\right).$$

For example,

$$\begin{aligned} |\pi_6^{n+1}(a_i)|_{\text{1-2}} &= \tfrac{5 \cdot 6^n}{12}\left(3 \cdot 6^{n+1} + 6 - 2\right) \\ &= 6^{n-1} \cdot (45 \cdot 6^n + 10), \\ |\pi_6^{n+1}(a_i)|_{\text{2-1}} &= 6^{n-1} \cdot (45 \cdot 6^n - 10). \end{aligned}$$

The Arshon morphisms, 1937

$\mathcal{A} = \{1, 2, \ldots, k\}$.

Let $w_1 = 1$. For $n \geqslant 1$, w_{n+1} is obtained by replacing the letters of w_n:

in odd positions	in even positions
$1 \to 123\ldots(k-1)k$	$1 \to k(k-1)\ldots321$
$2 \to 234\ldots(k-1)k1$	$2 \to 1k(k-1)\ldots432$
...	...
$k \to k12\ldots(k-2)(k-1)$	$k \to (k-1)(k-2)\ldots21k$

Then $w_2 = 123\ldots(k-1)k$ and each w_i is the initial subword of w_{i+1}, so $w = \lim_{i\to\infty} w_i$ is well defined.

Theorem. [Berstel 1979, SK, 2003] There does not exist a morphism, whose iteration defines the Arshon sequence for $k = 3$.

This is obvious that the Arshon sequences of even order are generated by a morphism.

Theorem. [Currie, 2002] No Arshon sequence of odd order can be generated by an iterated morphism.

The Arshon sequences, β_k^n, 1937

Let k be any even positive integer. For every i, $1 \leqslant i \leqslant k$, and for every positive integer n,

$$|\beta_k^{n+1}(a_i)|_{\text{1-2}} = \frac{k^{n-1}}{4}\left[k^{n+2}\cdot(k-1)+2k\right],$$

$$|\beta_k^{n+1}(a_i)|_{\text{2-1}} = \frac{k^{n-1}}{4}\left[k^{n+2}\cdot(k-1)-2k\right].$$

For example,

$$\begin{aligned}|\beta_6^{n+1}(a_i)|_{\text{1-2}} &= 6^{n-1}\cdot(45\cdot 6^n+3),\\ |\beta_6^{n+1}(a_i)|_{\text{2-1}} &= 6^{n-1}\cdot(45\cdot 6^n-3).\end{aligned}$$

More examples of morphisms satisfying the three conditions, but not linked with Thue-Morse sequence:

$$f : \begin{array}{rcl} a_1 & \mapsto & a_1 a_3 a_2 a_4 \\ a_2 & \mapsto & \varepsilon \\ a_3 & \mapsto & a_1 a_4 \\ a_4 & \mapsto & a_2 a_3 \end{array}$$

$$\begin{array}{rcccl} |f^{n+1}(a_1)|_{1\text{-}2} & = & |f^{n+1}(a_3 a_4)|_{1\text{-}2} & = & 3 \cdot 2^{n-1} \cdot (2^{n+1} + 1), \\ |f^{n+1}(a_1)|_{2\text{-}1} & = & |f^{n+1}(a_3 a_4)|_{2\text{-}1} & = & 3 \cdot 2^{n-1} \cdot (2^{n+1} - 1), \\ |f^{n+1}(a_2)|_{1\text{-}2} & = & |f^{n+1}(a_2)|_{2\text{-}1} & = & 0. \end{array}$$

More examples of morphisms satisfying the three conditions, but not linked with Thue-Morse sequence:

$$\begin{array}{rcl} h : a & \mapsto & aba\ cab\ cac\ bab\ cba\ cbc \\ b & \mapsto & aba\ cab\ cac\ bca\ bcb\ abc \\ c & \mapsto & aba\ cab\ cba\ cbc\ acb\ abc \end{array}$$

This morphism is square-free (Brandenburg, 1983)

For every $x \in \mathcal{A} = \{a < b < c\}$ and for every positive integer n,

$$|h^{n+1}(x)|_{1\text{-}2} = 6 \cdot 18^{n-1} \cdot (9 \cdot 18^{n+1} + 40),$$

$$|h^{n+1}(x)|_{2\text{-}1} = 6 \cdot 18^{n-1} \cdot (9 \cdot 18^{n+1} - 40).$$

Consecutive patterns and morphisms

The vector of rises of f^n is

$$R(f^n) = (|f^n(a_i)|_{12})_{1 \leqslant i \leqslant k}.$$

The vector of descents of f^n is

$$D(f^n) = (|f^n(a_i)|_{21})_{1 \leqslant i \leqslant k}.$$

The vector of squares of one letter of f^n is

$$R_2(f^n) = (|f^n(a_i)|_{11})_{1 \leqslant i \leqslant k}.$$

For $a_\ell \in \mathcal{A}$, $f(a_\ell) = a_{\ell_1} \ldots a_{\ell_{p_\ell}}$, and for all $n \geqslant M_f$ (where M_f is the mortality exponent), let $\ell'_1 \ldots \ell'_{p'_\ell}$ be the subsequence of $\ell_1 \ldots \ell_{p_\ell}$ such that $f^{n+1}(a_\ell) = f^n(a_{\ell'_1} \ldots a_{\ell'_{p'_\ell}})$ and $f^n(a_{\ell'_i}) \neq \varepsilon$, $1 \leqslant i \leqslant p'_\ell$.

Then

$$|f^{n+1}(a_\ell)|_{12} = \sum_{t=1}^{k} |f^n(a_t)|_{12} \cdot m_{1,t,\ell} + \sum_{i=1}^{p'_\ell - 1} C_n^{12}(\ell'_i, \ell'_{i+1}),$$

$$|f^{n+1}(a_\ell)|_{21} = \sum_{t=1}^{k} |f^n(a_t)|_{21} \cdot m_{1,t,\ell} + \sum_{i=1}^{p'_\ell - 1} C_n^{21}(\ell'_i, \ell'_{i+1}),$$

$$|f^{n+1}(a_\ell)|_{11} = \sum_{t=1}^{k} |f^n(a_t)|_{11} \cdot m_{1,t,\ell} + \sum_{i=1}^{p'_\ell - 1} C_n^{11}(\ell'_i, \ell'_{i+1}).$$

The Thue-Morse morphism

For any integer $n \geqslant 0$,

$$R(\mu^{2n}) = \left[\ \frac{4^n-1}{3} \quad \frac{4^n-1}{3}\ \right] = D(\mu^{2n}) = R_2(\mu^{2n})$$

$$R(\mu^{2n+1}) = \left[\ \frac{2(4^n-1)}{3}+1 \quad \frac{2(4^n-1)}{3}\ \right]$$

$$D(\mu^{2n+1}) = \left[\ \frac{2(4^n-1)}{3} \quad \frac{2(4^n-1)}{3}+1\ \right]$$

$$R_2(\mu^{2n+1}) = \left[\ \frac{2(4^n-1)}{3} \quad \frac{2(4^n-1)}{3}\ \right].$$

The Fibonacci morphism

For any integer $n \geqslant 1$,

$$
\begin{aligned}
R(\varphi^n) &= \left[\begin{array}{cc} F_{n-1} & F_{n-2} \end{array}\right] \\
D(\varphi^{2n}) &= \left[\begin{array}{cc} F_{2n-1} & F_{2n-2}-1 \end{array}\right] = R_2(\varphi^{2n+1}) \\
R_2(\varphi^{2n}) &= \left[\begin{array}{cc} F_{2n-2}-1 & F_{2n-3} \end{array}\right] = D(\varphi^{2n-1}).
\end{aligned}
$$

Erasing morphisms

$$\begin{aligned} f(a_1) &= a_1a_3a_2a_4 \\ f(a_2) &= \varepsilon \\ f(a_3) &= a_1a_4 \\ f(a_4) &= a_2a_3 \end{aligned}$$

One has $M_f = 1$.

For any integer $n \geqslant 1$, $R_2(f^n) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$\text{if } n \text{ is even} \begin{cases} R(f^n) &= \begin{bmatrix} 2^n & 0 & \frac{2^{n+1}+1}{3} & \frac{2^n-1}{3} \end{bmatrix} \\ D(f^n) &= \begin{bmatrix} 2^n - 1 & 0 & \frac{2^{n+1}-2}{3} & \frac{2^n-4}{3} \end{bmatrix}, \end{cases}$$

$$\text{if } n \text{ is odd} \begin{cases} R(f^n) &= \begin{bmatrix} 2^n & 0 & \frac{2^{n+1}-1}{3} & \frac{2^n+1}{3} \end{bmatrix} \\ D(f^n) &= \begin{bmatrix} 2^n - 1 & 0 & \frac{2^{n+1}-4}{3} & \frac{2^n-2}{3} \end{bmatrix}. \end{cases}$$

Erasing morphisms

$$
\begin{aligned}
g(a_1) &= a_1a_2a_4a_3 \\
g(a_2) &= a_3 \\
g(a_3) &= \varepsilon \\
g(a_4) &= a_1a_2a_4
\end{aligned}
$$

Here we have $M_g = 2$

$R(g) = \begin{bmatrix} 2 & 0 & 0 & 2 \end{bmatrix}$, $D(g) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $R_2(g) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$,
and, for any integer $n \geqslant 2$,

$$
\begin{aligned}
R(g^n) &= \begin{bmatrix} 2^n & 0 & 0 & 2^n \end{bmatrix} \\
D(g^n) &= \begin{bmatrix} 2^{n-1}+2^{n-2}-1 & 0 & 0 & 2^{n-1}+2^{n-2}-1 \end{bmatrix} \\
R_2(g^n) &= \begin{bmatrix} 2^{n-2} & 0 & 0 & 2^{n-2} \end{bmatrix}.
\end{aligned}
$$

1975 subword (factor) complexity (Ehrenfeucht, Lee, Rozenberg)

1976 Lempel-Ziv complexity (Lempel, Ziv)

1987 d-complexity (Iványi)

1995 palindrome (palindromic) complexity (Hof, Knill, Simon)

2000 arithmetical complexity (Avgustinovich, Fon-Der-Flaass, Frid)

2002 pattern complexity (Restivo, Salemi)

2002 maximal pattern complexity (Kamae, Zamboni)

Thank you for your attention!