Classifying Descents According to Equivalence mod k

Sergey Kitaev

Reykjavík University

Jeffrey Remmel

University of California, San Diego

The following papers are by S. Kitaev and J. Remmel:

- Classifying Descents According to Parity, Annals of Combinatorics, to appear.

- Classifying Descents According to Equivalence mod k, preprint.

$\sigma = 5\ 1\ 7\ 6\ 4\ 8\ 2\ 9\ 3$

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The descent set of σ: $Des(\sigma) = \{1, 3, 4, 6, 8\}$

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Eulerian numbers $A(n, k)$ count n-permutations with k descents.

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Eulerian numbers $A(n, k)$ count n-permutations with k descents

$A(n, k)$ are the coefficients of the Eulerian polynomials

$$A_n(t) = \sum_{\sigma \in S_n} t^{1+des(\sigma)}.$$

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Eulerian numbers $A(n, k)$ count n-permutations with k descents

$A(n, k)$ are the coefficients of the Eulerian polynomials

$$A_n(t) = \sum_{\sigma \in S_n} t^{1+des(\sigma)}.$$

The Eulerian numbers $A(n, k)$ satisfy the identity

$$\sum_{k \geqslant 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

$\sigma \;=\; \underline{5}\ 1\ \underline{7}\ \underline{6}\ 4\ \underline{8}\ 2\ \underline{9}\ 3$

The descent set of σ: $Des(\sigma) = \{1, 3, 4, 6, 8\}$

The parity of the **first** element of descents is fixed:

$\overleftarrow{Des}_{\mathsf{odd}}(\sigma) = \{1, 3, 8\}$

$\sigma \;=\; \underline{5}\ 1\ \underline{7}\ \underline{6}\ 4\ \underline{8}\ 2\ \underline{9}\ 3$

The descent set of σ: $Des(\sigma) = \{1, 3, 4, 6, 8\}$

The parity of the **first** element of descents is fixed:

$\overleftarrow{Des}_{\text{odd}}(\sigma) = \{1, 3, 8\}$

$\overleftarrow{Des}_{\text{even}}(\sigma) = \{4, 6\}$

$\sigma = \underline{5}\ 1\ \underline{7}\ \underline{6}\ 4\ \underline{8}\ 2\ \underline{9}\ 3$

The descent set of σ: $Des(\sigma) = \{1, 3, 4, 6, 8\}$

The parity of the **first** element of descents is fixed:

$\overleftarrow{Des}_{\text{odd}}(\sigma) = \{1, 3, 8\}$

$\overleftarrow{Des}_{\text{even}}(\sigma) = \{4, 6\}$

The parity of the **second** element of descents is fixed:

$\overrightarrow{Des}_{\text{odd}}(\sigma) = \{1, 8\}$

$\overrightarrow{Des}_{\text{even}}(\sigma) = \{3, 4, 6\}$

$\sigma \; = \; \underline{5}\ 1\ \underline{7}\ \underline{6}\ 4\ \underline{8}\ 2\ \underline{9}\ 3$

The descent set of σ: $Des(\sigma) = \{1, 3, 4, 6, 8\}$

The parity of the **first** element of descents is fixed:

$\overleftarrow{Des}_{\mathsf{odd}}(\sigma) = \{1, 3, 8\}$

$\overleftarrow{Des}_{\mathsf{even}}(\sigma) = \{4, 6\}$ $\quad \overleftarrow{Des}_{kN}(\sigma)$

The parity of the **second** element of descents is fixed:

$\overrightarrow{Des}_{\mathsf{odd}}(\sigma) = \{1, 8\}$

$\overrightarrow{Des}_{\mathsf{even}}(\sigma) = \{3, 4, 6\}$ $\quad \overrightarrow{Des}_{kN}(\sigma)$

$\sigma \;=\; \underline{5}\ 1\ \underline{7}\ \underline{6}\ 4\ \underline{8}\ 2\ \underline{9}\ 3$

The descent set of σ: $Des(\sigma) = \{1, 3, 4, 6, 8\}$

The parity of the **first** element of descents is fixed:

$\overleftarrow{Des}_{\mathsf{odd}}(\sigma) = \{1, 3, 8\}$

$\overleftarrow{Des}_{\mathsf{even}}(\sigma) = \{4, 6\}$ $\overleftarrow{Des}_{kN}(\sigma)$ $\overleftarrow{Des}_{3N}(\sigma) = \{4, 8\}$

The parity of the **second** element of descents is fixed:

$\overrightarrow{Des}_{\mathsf{odd}}(\sigma) = \{1, 8\}$

$\overrightarrow{Des}_{\mathsf{even}}(\sigma) = \{3, 4, 6\}$ $\overrightarrow{Des}_{kN}(\sigma)$ $\overrightarrow{Des}_{3N}(\sigma) = \{3, 8\}$

We define the following polynomials:

$$\overleftarrow{Des}_{\text{odd}}(\sigma) \quad \Rightarrow \quad M_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{odd}}(\sigma)} = \sum_{k=0}^{n} M_{k,n} x^k$$

$$\overleftarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{even}}(\sigma)} = \sum_{k=0}^{n} R_{k,n} x^k$$

$$\overleftarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$$

$$\overrightarrow{Des}_{\text{odd}}(\sigma) \quad \Rightarrow \quad Q_n(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{\text{odd}}(\sigma)} z^{\chi(\sigma_1 \text{ is odd})} = \sum_{k=0}^{n} \sum_{j=0}^{1} Q_{j,k,n} z^j x^k$$

$$\overrightarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad P_n(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{\text{even}}(\sigma)} z^{\chi(\sigma_1 \text{ is even})} = \sum_{k=0}^{n} \sum_{j=0}^{1} P_{j,k,n} z^j x^k$$

$$\overrightarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad B_n^{(k)}(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1 \in kN)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j$$

Main Goal: Study of the polynomials coefficients.

Outline of the talk:

- Overview of selected results;
- Combinatorial identities arising in the studies;
- Bijective proofs of certain results;
- Connection to the Genocchi numbers;
- Directions for further research.

$$\overleftarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{even}}(\sigma)} = \sum_{k=0}^{n} R_{k,n} x^k$$

Theorem. The polynomials $\{R_n(x)\}_{n \geqslant 1}$ are given by

1. $R_1(x) = 1$, $R_2(x) = 1 + x$, and for $n \geqslant 1$,

2. $R_{2n+1}(x) = (1-x)\frac{d}{dx}R_{2n}(x) + (1+2n)R_{2n}(x)$ and

3. $R_{2n+2}(x) = x(1-x)\frac{d}{dx}R_{2n+1}(x) + (1 + x(1+2n))R_{2n+1}(x)$.

$$\overleftarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{even}}(\sigma)} = \sum_{k=0}^{n} R_{k,n} x^k$$

Initial values of $R_n(x)$ are

$$\begin{aligned}
R_1(x) &= 1\\
R_2(x) &= 1 + x\\
R_3(x) &= 4 + 2x\\
R_4(x) &= 4 + 16x + 4x^2\\
R_5(x) &= 36 + 72x + 12x^2\\
R_6(x) &= 36 + 324x + 324x^2 + 36x^3\\
R_7(x) &= 576 + 2592x + 1728x^2 + 144x^3\\
R_8(x) &= 576 + 9216x + 20736x^2 + 9216x^3 + 576x^4
\end{aligned}$$

$$\overleftarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{even}}(\sigma)} = \sum_{k=0}^{n} R_{k,n} x^k$$

Theorem. We have

$$R_{k,2n} = \left(n! \binom{n}{k}\right)^2,$$

$$R_{k,2n+1} = \frac{1}{k+1} \left((n+1)! \binom{n}{k}\right)^2.$$

$$\boxed{\overleftarrow{Des}_{\text{odd}}(\sigma) \quad \Rightarrow \quad M_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{odd}}(\sigma)} = \sum_{k=0}^{n} M_{k,n} x^k}$$

Theorem. For all $0 \leqslant k \leqslant n$,

$$M_{k,2n} = \binom{n-1}{k}\binom{n+1}{k+1}(n!)^2,$$

$$M_{k,2n+1} = \binom{n}{k}\binom{n+1}{k}n!(n+1)!.$$

$$\overrightarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad \begin{aligned} P_n(x,z) &= \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{\text{even}}(\sigma)} z^{\chi(\sigma_1 \text{ is even})} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{1} P_{j,k,n} z^j x^k \end{aligned}$$

Theorem. The polynomials $\{P_n(x,z)\}_{n \geqslant 1}$ are given by

1. $P_1(x,z) = 1$, $P_2(x,z) = 1 + z$, and for all $n \geqslant 1$,

2. $P_{2n+1}(x,z) = x(1-x)\frac{\partial}{\partial x}P_{2n}(x,z) + x(1-z)\frac{\partial}{\partial z}P_{2n}(x,z) + (1+n(1+x))P_{2n}(x,z)$ and

3. $P_{2n+2}(x,z) = x(1-x)\frac{\partial}{\partial x}P_{2n+1}(x,z) + z(1-z)\frac{\partial}{\partial z}P_{2n+1}(x,z) + (1+z+n(1+x))P_{2n+1}(x,z)$.

$$\overrightarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad \begin{aligned} P_n(x,z) &= \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{\text{even}}(\sigma)} z^{\chi(\sigma_1 \text{ is even})} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{1} P_{j,k,n} z^j x^k \end{aligned}$$

The first few polynomials $P_n(x,z)$:

$P_1(x,z) = 1$
$P_2(x,z) = 1 + z$
$P_3(x,z) = 2 + 2z + 2x$
$P_4(x,z) = 4 + 8z + 8x + 4xz$
$P_5(x,z) = 12 + 24z + 48x + 24xz + 12x^2$
$P_6(x,z) = 36 + 108z + 216x + 216xz + 108x^2 + 36x^2z$
$P_7(x,z) = 144 + 432z + 1296x + 1296xz + 1296x^2 + 432x^2z + 144x^3$

$$\overrightarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad \begin{aligned} P_n(x,z) &= \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{\text{even}}(\sigma)} z^{\chi(\sigma_1 \text{ is even})} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{1} P_{j,k,n} z^j x^k \end{aligned}$$

Theorem. For all $0 \leqslant k \leqslant n$,

$$\begin{aligned} P_{1,k,2n} &= \binom{n-1}{k}\binom{n}{k+1}(n!)^2, \\ P_{0,k,2n} &= \binom{n-1}{k}\binom{n}{k}(n!)^2, \\ P_{0,k,2n+1} &= (n+1)\left(n!\binom{n}{k}\right)^2, \\ P_{1,k,2n+1} &= \frac{(n+1)(n-k)}{k+1}\left(n!\binom{n}{k}\right)^2. \end{aligned}$$

$$\overrightarrow{Des}_{\text{odd}}(\sigma) \quad \Rightarrow \quad \begin{aligned} Q_n(x,z) &= \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{\text{odd}}(\sigma)} z^{\chi(\sigma_1 \text{ is odd})} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{1} Q_{j,k,n} z^j x^k \end{aligned}$$

Theorem. The polynomials $\{Q_n(x,z)\}_{n \geqslant 1}$ are given by

1. $Q_1(x,z) = z$, $Q_2(x,z) = z + x$, and for all $n \geqslant 1$,

2. $Q_{2n+1}(x,z) = x(1-x)\frac{\partial}{\partial x}Q_{2n}(x,z) + z(1-z)\frac{\partial}{\partial z}Q_{2n}(x,z) + (z + n(1+x))Q_{2n}(x,z)$

3. $Q_{2n+2}(x,z) = x(1-x)\frac{\partial}{\partial x}Q_{2n+1}(x,z) + x(1-z)\frac{\partial}{\partial z}Q_{2n+1}(x,z) + (1+n)(1+x)Q_{2n+1}(x,z)$.

$$\overrightarrow{Des}_{\text{odd}}(\sigma) \Rightarrow \quad \begin{aligned} Q_n(x,z) &= \sum_{\sigma\in S_n} x^{\overrightarrow{des}_{\text{odd}}(\sigma)} z^{\chi(\sigma_1 \text{ is odd})} \\ &= \sum_{k=0}^{n}\sum_{j=0}^{1} Q_{j,k,n} z^j x^k \end{aligned}$$

Theorem. For all $0 \leqslant k \leqslant n$,

$$\begin{aligned} Q_{0,k,2n} &= \binom{n-1}{k-1}\binom{n}{k}(n!)^2, \\ Q_{0,k,2n+1} &= \frac{(n+1)(n-k+1)}{k}\left(\binom{n}{k-1}n!\right)^2, \\ Q_{1,k,2n} &= \binom{n-1}{k}\binom{n}{k}(n!)^2, \\ Q_{1,k,2n+1} &= n!(n+1)!\binom{n}{k}^2. \end{aligned}$$

$$\boxed{\overleftarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j}$$

Theorem. We have

1. $A_1^{(k)}(x) = 1$,

2. For $j = 1, \ldots, k-1$ and for $n \geqslant 0$, $A_{kn+j}^{(k)}(x) = (1-x)\frac{d}{dx}(A_{kn+j-1}^{(k)}(x))$ $+ (kn+j)A_{kn+j-1}^{(k)}(x)$, and

3. $A_{kn+k}^{(k)}(x) = (x-x^2)\frac{d}{dx}(A_{kn+k-1}^{(k)}(x)) + (1+x(kn+k-1))A_{kn+k-1}^{(k)}(x)$ for $n \geqslant 1$.

$$\overleftarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$$

Theorem. For $0 \leqslant j \leqslant k-1$ and $0 \leqslant s \leqslant n$,

$$A_{s,kn+j}^{(k)} = ((k-1)n+j)! \left[\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \times \prod_{i=0}^{n-1} (r+1+j+(k-1)i) \right].$$

$$\boxed{\overleftarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad A_n^{(k)}(x) = \sum_{\sigma\in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j}$$

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Corollary. [Two special cases of the Saalschütz's identity]

$$\binom{n}{s}^2 = \sum_{r=0}^{s} (-1)^{s-r} \binom{n+r}{r}^2 \binom{2n+1}{s-r};$$

$$\binom{n}{s}\binom{n+1}{s+1} = \sum_{r=0}^{s} (-1)^{s-r} \binom{n+r+1}{r} \binom{n+r+1}{r+1} \binom{2n+2}{s-r}.$$

$$\overleftarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$$

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$$A_{s,kn+j}^{(k)} = ((k-1)n+j)! \left[\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \times \prod_{i=0}^{n-1} (r+1+j+(k-1)i) \right].$$

$$A_{n-s,kn+j}^{(k)} = ((k-1)n+j)! \left[\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=1}^{n} (r+(k-1)i) \right].$$

Corollary. For $0 \leqslant j \leqslant k-1$ and $0 \leqslant s \leqslant n$,

$$\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r}\prod_{i=1}^{n}(r+(k-1)i) =$$

$$\sum_{r=0}^{n-s}(-1)^{n-s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}\prod_{i=0}^{n-1}(r+1+j+(k-1)i)$$

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$$\sum_{r=0}^{n-s}(-1)^{n-s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}\prod_{i=0}^{n-1}(r+1+j+(k-1)i)$$

$s = 0$:

$$(k-1)^n(n!) =$$

$$\sum_{r=0}^{n}(-1)^{n-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-r}\prod_{i=0}^{n-1}(r+1+j+(k-1)i).$$

$s = 1$:

$$((k-1)n+j+1)\prod_{i=0}^{n}(1+(k-1)i)-(kn+j+1)(k-1)^n(n!) =$$

$$\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-1-r}\prod_{i=0}^{n-1}(r+n+(k-1)i).$$

$$\overrightarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad \begin{aligned} B_n^{(k)}(x,z) &= \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1 \in kN)} \\ &= \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j \end{aligned}$$

Theorem. For any $k \geqslant 2$ and $n \geqslant 0$,

1. $B_1^{(k)}(x,z) = 1$,
2. $B_{kn+j+1}^{(k)}(x,z) = \left(x(1-x)\frac{\partial}{\partial x} + x(1-z)\frac{\partial}{\partial z} + nx + (1+(k-1)n+j)\right) \times (B_{kn+j}^{(k)}(x,z))$ for $0 \leqslant j \leqslant k-2$, and
3. $B_{kn+k}^{(k)}(x,z) = \left(x(1-x)\frac{\partial}{\partial x} + z(1-z)\frac{\partial}{\partial z} + nx + z + (k-1)(n+1)\right) \times (B_{kn+k-1}^{(k)}(x,z))$.

$$\overrightarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad \begin{aligned} B_n^{(k)}(x,z) &= \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1 \in kN)} \\ &= \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j \end{aligned}$$

Theorem. For all $n \geqslant 0$, $k \geqslant 2$, and $0 \leqslant j \leqslant k-1$,

$$B_{1,n-1-s,kn+j}^{(k)} =$$
$$((k-1)n+j)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j}{s-r} \times$$
$$\sum_{p=0}^{n-1} \left(\prod_{i=0}^{p-1} (j+(k-1)i) \right) \left(\prod_{i=p+1}^{n-1} (1+j+(k-1)i) \right).$$

$$\overrightarrow{Des}_{kN}(\sigma) \Rightarrow \begin{aligned} B_n^{(k)}(x) &= B_n^{(k)}(x,1) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} \\ &= \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} B_{j,n}^{(k)} x^j \end{aligned}$$

Theorem. For all $0 \leqslant s \leqslant n$, $k \geqslant 2$, and $0 \leqslant j \leqslant k-1$,

$$B_{n-s,kn+j}^{(k)} =$$

$$((k-1)n+j)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=0}^{n-1} (r+j+(k-1)i).$$

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Corollary. [Two special cases of the Saalschütz's identity]

$$\frac{n+1}{s+1}\binom{n}{s}^2 = \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{n+r}{r} \binom{n+r+1}{r} \binom{2n+2}{n-s-r};$$

$$\binom{n-1}{s}\binom{n+1}{s+1} = \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{n+r}{r} \binom{n+r-1}{r-1} \binom{2n+1}{n-s-r}.$$

$$\overrightarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad \begin{aligned} B_n^{(k)}(x) &= B_n^{(k)}(x,1) = \sum_{\sigma\in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} \\ &= \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} B_{j,n}^{(k)} x^j \end{aligned}$$

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$$B_{s,kn+j}^{(k)} =$$

$$((k-1)n+j)! \left[\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=1}^{n} (1+r+(k-1)i) \right].$$

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$$\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=1}^{n} (1+r+(k-1)i) =$$
$$\sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{n-s-r} \prod_{i=0}^{n-1} (r+j+(k-1)i).$$

Bijective proofs

$$\overleftarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{even}}(\sigma)} = \sum_{k=0}^{n} R_{k,n} x^k$$

$$R_2(x) = 1 + x$$

$$R_4(x) = 4 + 16x + 4x^2$$

$$R_6(x) = 36 + 324x + 324x^2 + 36x^3$$

$$R_8(x) = 576 + 9216x + 20736x^2 + 9216x^3 + 576x^4$$

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$$R_8(x) = 576 + 9216x + 20736x^2 + 9216x^3 + 576x^4$$

$$R_{k,2n} = \left(n!\binom{n}{k}\right)^2 = \left(n!\binom{n}{n-k}\right)^2 = R_{n-k,2n}$$

Bijective proofs

$$\overleftarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{even}}(\sigma)} = \sum_{k=0}^{n} R_{k,n} x^k$$

3 $\underline{8}$ $\underline{2}$ 1 5 $\underline{6}$ 4 7 — counted by $R_{3,8}$

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3 8 2 1 5 6 4 7 **9** — a dummy element is added

7 2 8 9 5 4 6 3 1 — the complement is taken

9 5 4 6 3 1 7 2 8 — a cyclic shift is made

5 4 $\underline{6}$ 3 1 7 2 8 — counted by $R_{4-3,8} = R_{1,8}$

Bijective proofs

$$\overleftarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{\text{even}}(\sigma)} = \sum_{k=0}^{n} R_{k,n} x^k$$

$$\overrightarrow{Des}_{\text{even}}(\sigma) \quad \Rightarrow \quad \begin{aligned} P_n(x,z) &= \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{\text{even}}(\sigma)} z^{\chi(\sigma_1 \text{ is even})} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{1} P_{j,k,n} z^j x^k \end{aligned}$$

$$R_{k,2n} = P_{0,k,2n} + P_{1,k-1,2n}$$

Bijective proofs

$$\overleftarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$$

$$\overrightarrow{Des}_{kN}(\sigma) \quad \Rightarrow \quad \begin{aligned} B_n^{(k)}(x,z) &= \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1 \in kN)} \\ &= \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j \end{aligned}$$

For all $k \geqslant 3$, $n \geqslant 0$, and $1 \leqslant j \leqslant \lfloor n/k \rfloor$,

$$A_{j,kn+k-2}^{(k)}(x) = A_{n-j,kn+k-2}^{(k)}(x).$$

$$A_{j,kn+k-2}^{(k)} = B_{0,j,kn+k-2}^{(k)} + B_{1,j-1,kn+k-2}^{(k)}.$$

Genocchi numbers can be defined by

$$\frac{2t}{e^t+1} = t + \sum_{n \geqslant 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}.$$

Dumont showed that the Genocchi number G_{2n} is the number of permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n+1}$ in S_{2n+1} such that

$$\begin{array}{ll} \sigma_i < \sigma_{i+1} & \text{if } \sigma_i \text{ is odd,} \\ \sigma_i > \sigma_{i+1} & \text{if } \sigma_i \text{ is even.} \end{array}$$

The first few Genocchi numbers are $1, 1, 3, 17, 155, 2073, \ldots$.

Superscripts e, o, or $*$ in a pattern are used to require that in an occurrence of the pattern in a permutation, the corresponding letters must be even, odd or either.

25314 has two occurrences of the pattern 2^*1^o (they are 53 and 31, both of them are occurrences of the pattern 2^o1^o), one occurrence of the pattern 1^o2^e (namely, 14), no occurrences of the pattern 1^o2^o, and no occurrences of the pattern 2^e1^*.

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We can state an alternative definition of the Genocchi numbers:

Definition. The Genocchi number G_{2n} is the number of permutations in S_{2n+1} avoiding simultaneously the patterns 1^e2^* and 2^o1^*.

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Conjecture. The number of permutations in S_{2n} avoiding simultaneously the patterns 2^*1^e and 2^e1^* is the Genocchi number G_{2n}.

Open problems

Give an inclusion-exclusion combinatorial argument to prove

$$A^{(k)}_{s,kn+j} = ((k-1)n+j)!\left[\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r}\times \prod_{i=0}^{n-1}(1+r+j+(k-1)i)\right].$$

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Open problems

A point to start might be the case $s = 0$ (avoidance) and $k = 2$, and understanding by inclusion-exclusion the fact that

$$A^{(2)}_{0,2n} = (n!)^2 \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{n-r} \binom{n+r}{n}^2;$$

$$A^{(2)}_{0,2n+1} = B^{(2)}_{0,2n+1} = (n+1)!n! \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+2}{n-r} \binom{n+r}{n} \binom{n+r+1}{r};$$

$$B^{(2)}_{0,2n} = (n!)^2 \sum_{r=0}^{n} (-1)^{n-r} \binom{2n+1}{n-r} \binom{n+r}{n} \binom{n+r-1}{n}.$$