Enumerating patterns in compositions

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Quick start

All partitions of 3:

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All compositions of 3:

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Quick start

All partitions of 3:

3	111
	21
	3

All compositions of 3:

3	111
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	3

Quick start

The number of all partitions of n:

$$p(n) = [x^n] \prod_{k=1}^{\infty} \frac{1}{1-x^k} \qquad \sim \frac{1}{4\sqrt{3}} \exp(\pi \sqrt{\frac{2n}{3}})$$

The number of all compositions of n:

$$2^{n-1}$$

The number of all compositions of n with k parts:

$$\binom{n-1}{k-1}$$

Quick start

All compositions of 3:

111 ← palindromic composition

21

12

3 ← palindromic composition

Quick start

A Carlitz composition is a composition in which no two consecutive parts are the same.

All compositions of 3:

111

21 $\leftarrow$ Carlitz composition

12 $\leftarrow$ Carlitz composition

3 $\leftarrow$ Carlitz composition

Quick start

Rise – a summand followed by a larger summand.

Drop – a summand followed by a smaller summand.

Level – a summand followed by itself.

All compositions of 3:

111 ← two consecutive levels

21 ← drop

12 ← rise

3

Research on compositions

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Compositions built on the elements of a set A.

- Alladi and Hoggatt (1975): $A = \{1, 2\}$
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- Chinn and Heubach (2003): $A = \mathbb{N} - \{k\}$
- Grimaldi (2000): $A = \{2k + 1, k \geqslant 0\}$
- Chinn, Grimaldi, and Heubach: $A = \mathbb{N}$

- Hoggatt and Bricknell (1975): compositions with parts in a general set A (generating function for the number of compositions and the number of parts is found)

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- Heubach and Mansour (2004):

  - The generating function for the number of Carlitz compositions of n with parts in A is given by

$$\frac{1}{1-\sum_{a\in A}\frac{x^a}{1+x^a}};$$

  - The generating function for the number of compositions of n with parts in A such that m parts are in B is given by

$$\frac{(\sum_{a\in B}x^a)^m}{(1-\sum_{a\in A\setminus B}x^a)^{m+1}}.$$

$C_n(A)$ – the number of compositions of n built on A;

$P_n(A)$ – the number of palindromic compositions of n built on A;

$C_A(x;y;r,\ell,d) = \sum_{n\geqslant 0}\sum_{\sigma\in C_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}$

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If $A = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$ then

$$P_A(x; y; r, \ell, d) = \frac{1 + \displaystyle\sum_{i=1}^{k} \frac{x^{a_i} y + x^{2a_i} y^2 (\ell - dr)}{1 - x^{2a_i} y^2 (\ell^2 - dr)}}{1 - \displaystyle\sum_{i=1}^{k} \frac{x^{2a_i} y^2 dr}{1 - x^{2a_i} y^2 (\ell^2 - dr)}}$$

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$$C_A(x;y;r,\ell,d) = \frac{1 + (1-d) \displaystyle\sum_{j=1}^{k} \frac{x^{a_j}y}{1 - x^{a_j}y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i}y(\ell - r)}{1 - x^{a_i}y(\ell - d)}}{1 - d \displaystyle\sum_{j=1}^{k} \frac{x^{a_j}y}{1 - x^{a_j}y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i}y(\ell - r)}{1 - x^{a_i}y(\ell - d)}}$$

Rises in compositions: consider $\frac{\partial}{\partial r} C_A(x; y; r, 1, 1)$ at $r = 1$ to get

$$\sum_{n \geqslant 0} \sum_{\sigma \in C_n^A} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)} =$$

$$\left(\sum_{k \geqslant j > i \geqslant 1} x^{a_i + a_j} \right) \sum_{m \geqslant 0} (m+1) \left(\sum_{j=1}^{k} x^{a_j} \right)^m y^{m+2}$$

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Levels in compositions: consider $\frac{\partial}{\partial \ell} C_A(x; y; 1, \ell, 1)$ at $\ell = 1$ to get

$$\sum_{n \geqslant 0} \sum_{\sigma \in C_n^A} \text{levels}(\sigma) x^n y^{\text{parts}(\sigma)} =$$

$$\left(\sum_{j=1}^{k} x^{2a_j} \right) \sum_{m \geqslant 0} (m+1) \left(\sum_{j=1}^{k} x^{a_j} \right)^m y^{m+2}$$

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Generating function for Carlitz compositions is given by $C_A(x; y; r, 0, d)$.

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To answer such questions we introduce segmented patterns in compositions.

13183316 – a composition of 25

1**3183**316 – an occurrence of the segmented pattern 2132

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A level followed by a rise is an occurrence of the pattern 112.

Level – 11; Drop – 21; Rise – 12

13183316 – a composition of 25

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Level – 11; Drop – 21; Rise – 12

A rise followed by a rise corresponds to the pattern 123.

A rise followed by a drop is either 132 or 231 =segmented partially ordered pattern 13'2 where 1<3' and 2<3' are the only relations on the poset.

A poset is built on 1, 1’ and 2’ with the only relation 1’<2’;

All linear extensions of 11’2’ are 123, 213 and 312.

Occurrences of 11’2’ in 31254 are 312 and 125.

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If μ_k is the number of k's in a composition v, then $\mu = (\mu_1, \ldots, \mu_j)$ is the content vector of v (j is the largest letter of v).

Example: $\mu = (2, 0, 1, 2, 1)$ is the content vector of 411345.

Composition 253 of 10 is order isomorphic to compositions 154, 163 and 172 of 10.

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The generating function for the number of compositions of n that are order isomorphic to pattern v is

$$\mathcal{P}_v(x) = \prod_{k=1}^{j} \frac{x^{m_k}}{1 - x^{m_k}},$$

where $m_k = \mu_{j-k+1} + \cdots + \mu_j$ for $1 \leqslant k \leqslant j$ and $(\mu_1, \ldots, \mu_j)$ is the content vector of v.

$w = w_1 w_2 \cdots w_m$ – a segmented pattern with m parts;

$c_w(n, \ell, s)$ – the number of occurrences of w among compositions of n with $\ell + m$ parts such that the sum of the parts preceding the occurrence is s.

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Theorem. [Kitaev, McAllister and Petersen, 2005]
For a segmented partially ordered pattern w, we have

$$\sum_{n,\ell,s \in \mathbb{N}} c_w(n, \ell, s) x^n y^\ell z^s = \frac{(1-x)(1-xz)}{(1-x-xy)(1-xz-xyz)} \sum_v \mathcal{P}_v(x)$$

where the sum to the right is over all linear extensions v of w.

Corollary. [Kitaev, McAllister and Petersen, 2005]
Given a segmented pattern w, the number of occurrences of w among compositions of n is equal to

$$[x^n]\frac{(1-x)^2}{(1-2x)^2}\sum_v \mathcal{P}_v(x),$$

where the sum is over all linear extensions v of w.

Corollary. [Kitaev, McAllister and Petersen, 2005]
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where the sum is over all linear extensions v of w.

Example. We compute the number of occurrences of m levels immediately followed by a rise (corresponding pattern: $w = \underbrace{1\cdots1}_{m+1}2$). The content vector of w is $\mu = (m+1, 1)$, so we have

$$\mathcal{P}_w(x) = \frac{x^{m+3}}{(1-x)(1-x^{m+2})}$$

and the number of occurrences of w among all compositions of n is

$$[x^n]\frac{(1-x)^2 x^{m+3}}{(1-2x)^2(1-x)(1-x^{m+2})}.$$

Two corollaries of the theorem above:

- The generating function for the expected number of occurrences of w in a randomly selected composition of n is:

$$\frac{(2-x)^2}{2(1-x)^2}\sum_v \mathcal{P}_v(x/2),$$

where the sum is over all linear extensions v of w.

- Given $n \in \mathbb{N}$ and $k \in [n-1]$, the number of k-parts among all compositions of n is $2^{n-k-2}(n-k+3)$.

Encoding by restricted permutations

Theorem. [Chinn, Grimaldi, and Heubach, 2003]
The number of k-parts in palindromic compositions of $2(n-1)$ is $(n-k+1)2^{n-k-1}$ when k is odd.

One can prove this result combinatorially by finding a bijection between the k-parts in the palindromic compositions and the permutations $w_1 w_2 \cdots w_{n-k+1}$ of $\{1, 2, \ldots, n-k+1\}$ such that, for some $\ell \in \{2, 3, \ldots, n\}$, $w_2 > w_3 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k+1}$.

The number of all palindromic compositions of n is $P_n = 2^{\lfloor n/2 \rfloor}$ and thus the number C_n of all compositions is

$$C_n = \begin{cases} \frac{1}{2} P_n^2 & \text{if } n = 2k \\ P_n^2 & \text{if } n = 2k+1 \end{cases}$$

Conjecture. For each integer n there exists at least one tiling of the $P_n \times nP_n$ rectangle using

- the C_n bargraphs corresponding to the compositions of n, if n is odd;

- the C_n bargraphs corresponding to the compositions of n used twice, if n is even;

Each bargraph can be rotated.