Uniquely k-determined permutations

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Permutation 253641 contains two occurrences of the consecutive pattern 132: <u>253</u>164 and 25<u>364</u>1

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More general question: Find joint distribution of patterns from a given set of consecutive patterns.

Approaches to study consecutive patterns:

1. Direct combinatorial arguments;

2. Method of inclusion-exclusion;

3. Tree representations of permutations;

4. Spectral theory of integral operators on $L^2([0,1]^k)$;

   ...

n. Considering the graph of patterns overlaps.

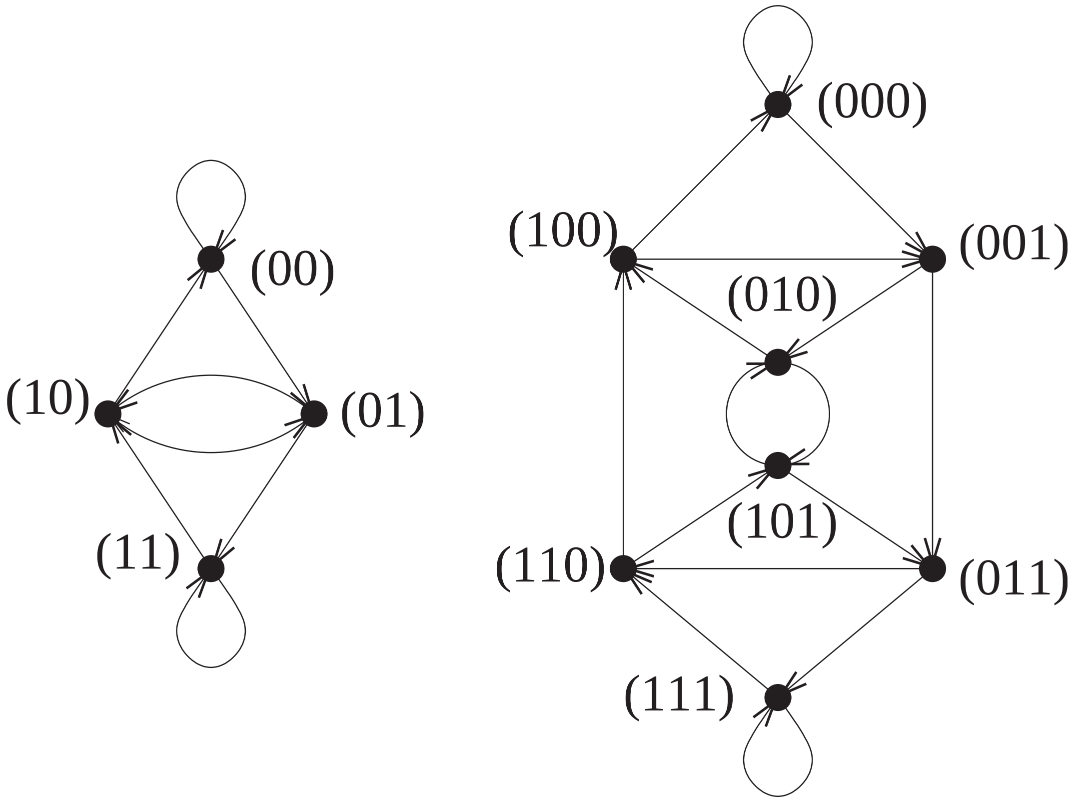
1. Direct combinatorial argument:
   $A_n(123, 321, 132) = (n-1)!! + (n-2)!!$ (SK)

2. Method of inclusion-exclusion: Generating function for $A_n(12543)$ is $\left(1 - x + \sum_{i \geqslant 1} \frac{(-1)^{i+1} x^{4i+1}}{(4i+1)!} \prod_{j=2}^{i} \binom{4j-2}{2}\right)^{-1}$ (SK)

3. Tree representations of permutations: Bivariate GF for distribution of 132 is $\left(1 - \int_0^z \exp((u-1)t^2/2)dt\right)^{-1}$ (Elizalde, Noy)

4. Spectral theory of integral operators on $L^2([0,1]^k)$:
$$\frac{A_n(213)}{n!} = \lambda_0^{n+1} \exp\left(\frac{1}{2\lambda_0^2}\right) + \mathcal{O}\left(\left(\frac{1}{\sqrt{2}}\right)^n\right)$$
   where $\lambda_0 = 0.7839769312\ldots$ (Ehrenborg, SK, Perry)

The de Bruijn graphs for the alphabet $A = \{0, 1\}$ and $n = 2, 3$:



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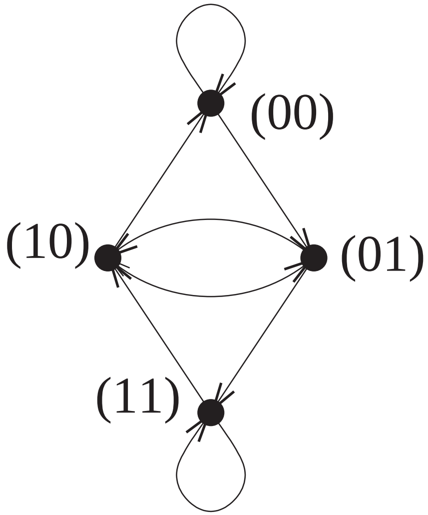

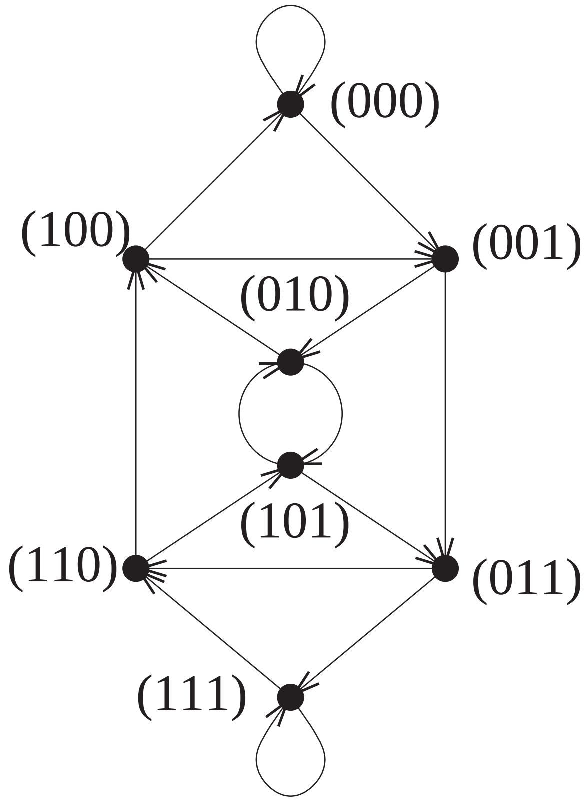

Graph of patterns overlaps: permutations instead of binary words.

For example, $2\underline{314} \to \underline{213}4$ is an arc in a graph of patterns overlaps, since 314 is order isomorphic to 213.

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Observation: For any n-permutation, there is a (unique) path in $\mathcal{P}_k$ of length $n - k + 1$ corresponding to it (assuming $n \geqslant k$).

Example: $k = 3$; to 13542 there corresponds the path $123 \to 132 \to 321$ in $\mathcal{P}_3$.

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Here a verbal description of our approach comes ...

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Uniquely k-determined permutations are those that can be reconstructed uniquely from the path corresponding to them.

Example: $12\dots n$ is uniquely k-determined for any $k \geqslant 2$; no n-permutation, $n \geqslant 2$, is uniquely 1-determined; each n-permutation is uniquely n-determined.

A few questions to ask:

1. Given a permutation, is it uniquely k-determined?

2. How many uniquely k-determined permutations are there? Is the generating function for the number of these permutations rational?

3. Suppose k is fixed; does there exist a finite set of prohibitions describing the uniquely k-determined permutations?

4. What is the structure of the uniquely k-determined permutations?

First criterion on unique k-determinability

Suppose $\pi = \pi_1\pi_2 \ldots \pi_n$ is a permutation and $i < j$. The distance $d_\pi(\pi_i, \pi_j) = d_\pi(\pi_j, \pi_i)$ between π_i and π_j is $j - i$. For example, $d_{25\underline{3}1\underline{6}4}(3, 6) = d_{25\underline{3}1\underline{6}4}(6, 3) = 2$.

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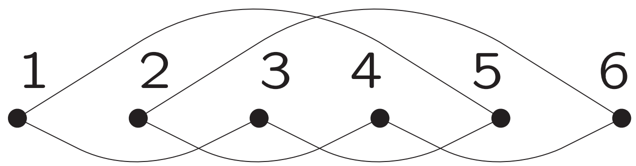
Coming back to $1\underline{3}54\underline{2}$ we see why it isn't uniquely 3-determined: $d_{13542}(2,3) = 3 = k$.

Second criterion on unique k-determinability

$V = \{1, 2, \ldots, n\}$ and M is a subset of V. A path-scheme $P(n, M)$ is a graph $G = (V, E)$, where the edge set E is $\{(x, y) \mid |x-y| \in M\}$. For example, $P(6, \{2, 4\})$ is



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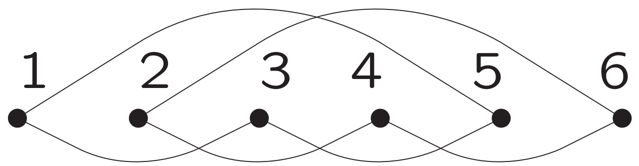

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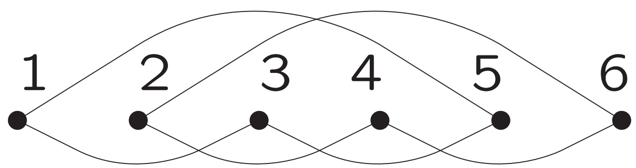

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Theorem. Let Φ be a map that sends a uniquely k-determined n-permutation π to the directed hamiltonian path in $\mathcal{G}_{n,n}$ corresponding to π^{-1}. Φ is a bijection between the set of all uniquely k-determined n-permutations and the set of all directed hamiltonian paths in $\mathcal{G}_{k,n}$.

A quick checking of whether an n-permutation π is uniquely k-determined or not: consider the $n-1$ differences of the adjacent elements in π^{-1} to see whether at least one of those differences exceeds $k-1$ or not.

The number of uniquely k-determined n-permutations, $n \geqslant 1$:

$k=2$	1, 2, 2, 2, 2, 2, 2, 2, 2, ...
$k=3$	1, 2, 6, 12, 20, 34, 56, 88, 136, ...
$k=4$	1, 2, 6, 24, 72, 180, 428, 1042, 2512, ...
$k=5$	1, 2, 6, 24, 120, 480, 1632, 5124, 15860, ...
$k=6$	1, 2, 6, 24, 120, 720, 3600, 15600, 61872, ...
$k=7$	1, 2, 6, 24, 120, 720, 5040, 30240, 159840, ...
$k=8$	1, 2, 6, 24, 120, 720, 5040, 40320, 282240, ...

The sequence corresponding to the case $k=3$ appears in Sloane, where we learn that the inverses to the uniquely 3-determined permutations are called key permutations.

Theorem. We have, for the number $A_{k,n}$ of uniquely k-determined n-permutations,

$$2((k-1)!)^{\lfloor n/k \rfloor} < A_{k,n} < 2(2(k-1))^n.$$

Prohibitions giving uniquely k-determined permutations

Let $|X|$ be the number of elements in X.

The set of uniquely k-determined n-permutations can be described by prohibiting patterns $xX(x+1)$ and $(x+1)Xx$, where X is a permutation on $\{1, 2, \ldots, |X|+2\} - \{x, x+1\}$, $|X| \geqslant k-1$, and $1 \leqslant x < n$.

We collect all such patterns in $\mathcal{L}_{k,n}$; also, let $\mathcal{L}_k = \cup_{n \geqslant 0} \mathcal{L}_{k,n}$.

Prohibitions giving uniquely k-determined permutations

A prohibited pattern $X = aYb$ from $\mathcal{L}_k$, where a and b are some consecutive elements, is called irreducible if the patterns of Yb and aY are not prohibited, that is, the patterns of Yb and aY are uniquely k-determined permutations.

Let $\mathcal{L}_k$ consists only of irreducible prohibited patterns.

Prohibitions giving uniquely k-determined permutations

Theorem. Suppose k is fixed. The number of (irreducible) prohibitions in $\mathcal{L}_k$ is finite. Moreover, the longest prohibited patterns in $\mathcal{L}_k$ are of length $2k-1$.

Here it comes a verbal description of how we use the theorem above and the graph of patterns overlaps $\mathcal{P}_{2k-1}$ to apply the transfer matrix method ...

Theorem. The generating function $A_k(x) = \sum_{n\geqslant 0} A_{k,n}x^n$ for the number of uniquely k-determined permutations is rational.

An n-permutation is crucial if it is uniquely k-determined, but adjoining any letter to the right of it, and thus creating an $(n+1)$-permutation, leads to a non-uniquely k-determined permutation.

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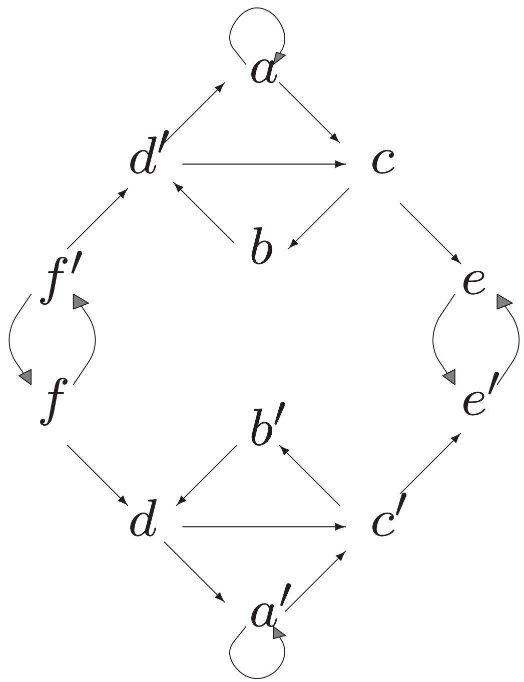
Theorem. There are no crucial permutations.

The case $k = 3$

Suppose w' denotes the complement to an n-permutation w. All uniquely 3-determined 4-permutations:

$$
\begin{array}{ll}
a = 1234 & a' = 4321 \\
b = 1324 & b' = 4231 \\
c = 1243 & c' = 4312 \\
d = 3421 & d' = 2134 \\
e = 1423 & e' = 4132 \\
f = 3241 & f' = 2314
\end{array}
$$

The case $k = 3$



$$A_3(x) = \sum_{n \geqslant 0} A_{3,n} x^n = \frac{1 - 2x + 2x^2 + x^3 - x^5 + x^6}{(1 - x - x^3)(1 - x)^2}.$$

Open problems

Any n-permutation is uniquely n-determined, whereas for $n \geqslant 2$ no n-permutation is uniquely 1-determined. Moreover, for any $n \geqslant 2$ there are exactly two uniquely 2-determined permutations, namely the monotone permutations.

Index $IR(\pi)$ of reconstructibility is the minimal integer k such that the permutation π is uniquely k-determined.

Problem 1. Describe the distribution of $IR(\pi)$ among all n-permutations.

Open problems

Problem 2. Study the set of uniquely k-determined permutations in the case when a set of nodes is removed from $\mathcal{P}_k$, that is, when some of patterns of length k are prohibited.

Open problems

An n-permutation π is m-k-determined, $m, k \geqslant 1$, if there are exactly m (different) n-permutations having the same path in $\mathcal{P}_k$ as π has. In particular, the uniquely k-determined permutations correspond to the case $m = 1$.

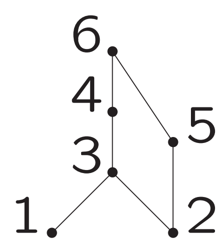
Problem 3. Find the number of m-k-determined n-permutations.

Open problems

An n-permutation π is m-k-determined, $m, k \geqslant 1$, if there are exactly m (different) n-permutations having the same path in $\mathcal{P}_k$ as π has.

Problem 3. Find the number of m-k-determined n-permutations.

Problem 3 is directly related to finding the number of linear extensions of a poset. Indeed, to any path w in $\mathcal{P}_k$ there naturally corresponds a poset $\mathcal{W}$. In particular, any factor of length k in w consists of comparable to each other elements in $\mathcal{W}$. If $k = 3$ and $w = 134265$ (7-3-determined) then $\mathcal{W}$ is the following poset:



Open problems

Recall that $\mathcal{L}_k$ is a set of irreducible prohibited patterns giving all uniquely k-determined permutations.

Problem 4. Describe the structure of $\mathcal{L}_k$. Is there a nice way to generate $\mathcal{L}_k$? How many elements does $\mathcal{L}_k$ have?

Thank you for your attention!

Questions?