Patterns and their generalizations

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Occurrences of the “classical" pattern 132 in 13524:

1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4, 1 3 5 2 4

Occurrences of the “classical" pattern 132 in 13524:

1 3 5 2 4,   1 3 5 2 4,   1 3 5 2 4,   1 3 5 2 4

A generalized pattern is a pattern that allows the requirement that two adjacent letters in the pattern must be adjacent in the permutation.

Pattern	Occurrences in 1342
1-3-2	1 3 4 2, 1 3 4 2
1-32	1 3 4 2
132	no occurrences

1969 D. Knuth: The Art of computer programming, vol. I

1985 R. Simion, F. Schmidt: Restricted permutations, European J. Combin. **6**, no. 4, 383–406.

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1992 Present: Explosive growth (several hundreds papers appeared)

2002 H. Wilf: The patterns of permutations, DM **257**, 575–583.

2003 S. Kitaev, T. Mansour: Survey of certain pattern problems

2004 M. Bóna: Combinatorics of Permutations, xiv+383 pp.

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2004 M. Atkinson: Permutation Patterns Home page http://www.cs.otago.ac.nz/staffpriv/mike/PPPages/PPhome.html

Permutation Patterns:
Classical patterns: Knuth, 1969 Generalized patterns: Babson and Steingrímsson, 2000 Partially ordered patterns: Kitaev, 2001

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Patterns in matrices:	Kitaev, Mansour and Vella, 2003
Patterns in n-dimensional objects:	Kitaev and Robbins, 2004

Patterns in even (odd) permutations:	Simion and Schmidt, 1985
Colored patterns in colored permutations:	Mansour, 2001
Signed patterns in signed permutations:	Mansour and West, 2002
Patterns with respect to parity:	Kitaev and Remmel, 2005

Let R be a set of patterns.

Let $S_n(p)$ be the set of all permutations in S_n which avoid the pattern p.

Then $S_n(R) = \bigcap_{p \in R} S_n(p)$.

An extreme case is $S_n(\emptyset) = S_n$ for all $n \geqslant 1$.

$N_n(R)$ is the number of elements of $S_n(R)$.

Questions about $S_n(R)$:

1. Formula for $N_n(R)$;

2. Generating function for $N_n(R)$, that is, $f_R(x) = \sum_i N_i(R) x^i$;

3. Relations to other combinatorial structures;

4. Is $S_n(R) = S_n(R')$ for all n?

In this case R and R' are said to be from the same Wilf class.

5. P-recursiveness of $N_n(R)$;

A function $f : \mathbf{N} \to \mathbf{C}$ is called P-recursive if there exist polynomials P_0, $P_1, \ldots, P_k \in \mathbf{C}[n]$, so that for all $n \in \mathbf{N}$

$$P_k(n) f(n+k) + P_{k-1}(n) f(n+k-1) + \cdots + P_0(n) f(n) = 0.$$

Theorem. [Knuth] For all $n \geqslant 1$, and for all classical patterns $p \in S_3$, $N_n(p)$ is given by the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

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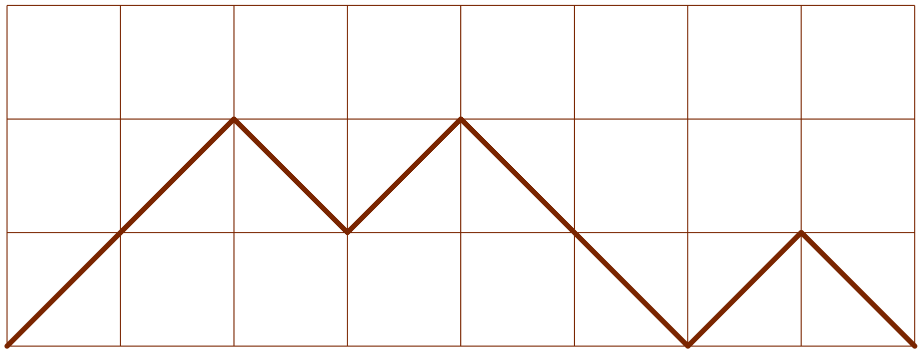
Dyck paths



pattern p	formula for $N_n(p)$	P-recursive
1-2-3-4 4-3-2-1	$(\star)$ (Gessel)	yes (Zeilberger)
1-3-4-2 2-4-3-1 3-1-2-4 4-2-1-3	$(\star\star)$ (Bóna)	yes (Bóna)
1-3-2-4 4-2-3-1	open	open

$$(\star) = 2\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2\frac{3k^2+2k+1-n-2kn}{(k+1)^2(k+2)(n-k+1)}$$

$$(\star\star) = \frac{7n^2-3n-2}{2}\cdot(-1)^{n-1}+3\sum_{i=2}^{n}2^{i+1}\cdot\frac{(2i-4)!}{i!(i-2)!}\binom{n-i+2}{2}(-1)^{n-i}$$

Theorem. [Regev] For all n, N_n(1-2-$\cdots$-k) asymptotically equals

$$\lambda_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}.$$

Here

$$\lambda_k = {\gamma_k}^2 \int_{x_1 \geqslant} \int_{x_2 \geqslant} \cdots \int_{\geqslant x_k} [D(x_1, x_2, \ldots, x_k) \cdot e^{-(k/2)x^2}]^2 \, dx_1 dx_2 \ldots dx_k,$$

where $D(x_1, x_2, \ldots, x_k) = \prod_{i<j} (x_i - x_j)$ and $\gamma_k = (1/\sqrt{2\pi})^{k-1} \cdot k^{k^2/2}$.

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Theorem. [Marcus and Tardos] For every permutation pattern p, there is a constant $c = c(p) < \infty$ such that for all n $N_n(p) < c^n$. [This was the famous Stanley-Wilf Conjecture]

Multi-avoidance of classical patterns

For avoiding a pair of classical 3-patterns, we have 3 Wilf classes with $N_n(p)$ given by 2^{n-1}, $\binom{n}{2}+1$ and 0 (Simion and Schmidt).

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restrictions	formula	author
1-2-3,4-3-2-1	0	West
1-2-3,3-4-2-1	$\binom{n}{4}+2\binom{n}{3}+n$	West
1-3-2,4-3-2-1	$\binom{n}{4}+\binom{n+1}{4}+\binom{n}{2}+1$	West
1-2-3,4-2-3-1	$\binom{n}{5}+2\binom{n}{4}+\binom{n}{3}+\binom{n}{2}+1$	West
1-2-3,3-2-4-1	$3\cdot 2^{n-1}-\binom{n+1}{2}-1$	West
1-2-3,3-4-1-2	$2^{n+1}-\binom{n+1}{3}-2n-1$	Stanley
1-3-2,4-2-3-1	$1+(n-1)2^{n-2}$	Guibert
1-3-2,3-4-2-1	$1+(n-1)2^{n-2}$	West
1-3-2,3-2-1-4	GF: $\frac{(1-x)^3}{1-4x+5x^2-3x^3}$	West

The following were given by West:

restrictions	restrictions	formula
1-2-3,2-1-4-3 1-2-3,2-4-1-3 1-3-2,2-3-1-4 1-3-2,2-3-4-1 3-1-2,2-3-1-4 1-3-2,3-4-1-2 3-1-2,1-4-3-2	3-1-2,1-3-4-2 3-1-2,3-2-4-1 3-1-2,3-2-1-4 1-2-3,3-2-1-4 3-1-2,4-3-2-1 3-1-2,3-4-2-1 1-3-2,3-2-4-1	F_{2n} (Fibonacci number)
3-1-4-2,2-4-1-3	4-1-3-2,4-2-3-1	GF: $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$

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restrictions	restrictions	formula
1-2-3,2-1-4-3 1-2-3,2-4-1-3 1-3-2,2-3-1-4 1-3-2,2-3-4-1 3-1-2,2-3-1-4 1-3-2,3-4-1-2 3-1-2,1-4-3-2	3-1-2,1-3-4-2 3-1-2,3-2-4-1 3-1-2,3-2-1-4 1-2-3,3-2-1-4 3-1-2,4-3-2-1 3-1-2,3-4-2-1 1-3-2,3-2-4-1	F_{2n} (Fibonacci number)
3-1-4-2,2-4-1-3	4-1-3-2,4-2-3-1	GF: $\frac{1-x-\sqrt{1-6x+x^2}}{2x}$

Theorem. [Simion and Schmidt] For every $n \geqslant 1$,

$$N_n(1\text{-}2\text{-}3, 1\text{-}3\text{-}2, 2\text{-}1\text{-}3) = F_{n+1},$$

where F_n is the n-th Fibonacci number.

Generalized patterns

The following were given by Claesson

Generalized patterns	Related combinatorial objects
2-31	Dyck paths (Catalan numbers)
1-23	Partitions (Bell numbers)
1-23, 12-3	Non-overlapping partitions (Bessel numbers)
1-23, 1-32	Involutions
1-23, 13-2	Motzkin paths

Claesson and Mansour provided complete solution for the number of permutations avoiding a pair of type x-yz or xy-z. Out of $\binom{12}{2} = 66$ pairs there are 21 symmetry classes and 10 Wilf classes.

The following were given by Kitaev:

Restrictions	**Formula**
123, 321, 132, 213	$2C_k$, if $n = 2k+1$ $C_k + C_{k-1}$, if $n = 2k$ (C_k – Catalan number)
123, 132, 213	$\binom{n}{\lfloor n/2 \rfloor}$
123, 132, 231	n
132, 213, 312	$1 + 2^{n-2}$
123, 132, 312	Recursive Formula
123, 321, 231	$(n-1)!! + (n-2)!!$
123, 231, 312	EGF: $1 + x(\sec(x) + \tan(x))$ (with Mansour)
132, 213	Recursive Formula (with Mansour)
123, 321	$2E_n$, where E_n is the n-th Euler number
132, 231	2^{n-1}

Beginning (or ending) with a k-pattern $\equiv$ Avoidance of $k! - 1$ generalized patterns simultaneously

For example, beginning with the pattern 123 is equivalent to the simultaneous avoidance of the patterns [132), [213), [231), [312) and [321) in the Babson-Steingrímsson notation.

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We can consider beginning with a pattern and avoiding another pattern.

Motivation: Relations to certain classes of labelled trees.

avoid	begin	EGF
123	$12\cdots k$	$\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k=1$ $\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}-\frac{1}{2}-\frac{\sqrt{3}}{2}\tan(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})$, if $k=2$ 0, if $k\geqslant 3$
123	$k\cdots 21$ ★	$\frac{\sqrt{3}}{2}\frac{e^{x/2}}{\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k=1$ $\frac{e^{x/2}\int_0^x e^{-t/2}t^{k-1}\sin(\frac{\sqrt{3}}{2}t+\frac{\pi}{6}))\ dt}{(k-1)!\cos(\frac{\sqrt{3}}{2}x+\frac{\pi}{6})}$, if $k\geqslant 2$

★ indicates that there is a generalization in this case

avoid	begin	EGF
132	$12\cdots k$	$(1-\int_0^x e^{-t^2/2}\,dt)^{-1}$, if $k=1$ $e^{-x^2/2}(1-\int_0^x e^{-t^2/2}\,dt)^{-1}-x-1$, if $k=2$ $(1-\int_0^x e^{-t^2/2}\,dt)^{-1}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_2}(e^{-t_1^2/2}-$ $(t_1+1)(1-\int_0^{t_1} e^{-t^2/2}\,dt))dt_1dt_2\cdots dt_{k-2}$, if $k\geqslant 3$
132	$k\cdots 21$ ★	$(1-\int_0^x e^{-t^2/2}\,dt)^{-1}$, if $k=1$ $\frac{1}{(k-1)!(1-\int_0^x e^{-t^2/2}\,dt)}\int_0^x t^{k-1}e^{-t^2/2}\,dt$, if $k\geqslant 2$

avoid	begin	EGF
213	$12\cdots k$	$(1-\int_0^x e^{-t^2/2}\,dt)^{-1}$, if $k=1$ $\int_0^x\int_0^t \frac{s^{k-2}e^{T(t)-T(s)}}{(k-2)!(1-\int_0^t e^{-m^2/2}dm)}\,dsdt$, if $k\geqslant 2$, where $T(x)=-x^2/2+\int_0^x \frac{e^{-t^2/2}}{1-\int_0^t e^{-s^2/2}ds}\,dt$
213	$k\cdots 21$	$(1-\int_0^x e^{-t^2/2}\,dt)^{-1}$, if $k=1$ $\frac{-x^{k-1}}{(k-1)!}+\sum_{n=0}^{k-2}\int_0^x\int_0^{t_n}\cdots\int_0^{t_1}\frac{C_{k-n}(t)+\delta_{n,k-2}}{1-\int_0^t e^{-m^2/2}dm}dtdt_1\cdots dt_n$, if $k\geqslant 2$, where $C_k(x)=e^{T(x)}\int_0^x\int_0^{t_{k-2}}\cdots\int_0^{t_1}e^{-T(t)}$. $\left(\frac{e^{-t^2/2}}{1-\int_0^t e^{-m^2/2}dm}-t-1\right)dtdt_1\cdots dt_{k-2}$

Kitaev and Mansour considered permutations that avoid a pattern of the form x-yz or xy-z and begin with one of the patterns $12\cdots k$, $k(k-1)\cdots 1$, $23\cdots k1$ and $(k-1)(k-2)\cdots 1k$.

Kitaev and Mansour considered permutations that avoid a pattern of the form (x-yz) or (xy-z) and begin with one of the patterns $12\cdots k$, $k(k-1)\cdots 1$, $23\cdots k1$ and $(k-1)(k-2)\cdots 1k$.

Generating functions that appear in this context

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

$$x^k C^2(x), \qquad x^k C^3(x), \qquad x^k C^{k+1}(x)$$

Exponential generating functions that appear

$$\frac{e^{e^x}}{(k-1)!}\int_0^x t^{k-1}e^{-e^t+t}\ dt \qquad e^{e^x}\int_0^x e^{-e^t}\sum_{n\geqslant k-1}\frac{t^n}{n!}\ dt$$

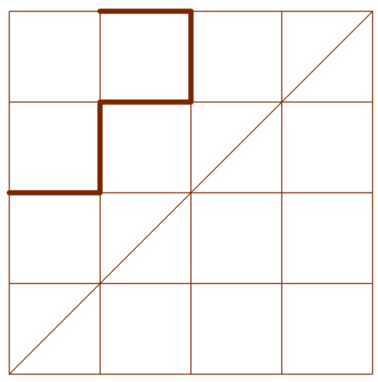
$$e^{e^x}\int_0^x e^{-e^t}(e^t-1)\ dt \qquad e^{e^x}\int_0^x\int_0^t \frac{r^{k-2}}{(k-2)!}e^{r-e^t}\ dr dt$$

$$x^2\sum_{j=0}^{k}(1-jx)^{-1}\sum_{d\geqslant 0}\frac{x^d}{(1-x)(1-2x)\dots(1-dx)}$$

Next step — counting the number of permutations that avoid a 3-pattern, begin and end with some given patterns

Some of these numbers are in the Sloane On-Line Encyclopedia

Example. For $n \geqslant 2$, the number of integer lattice paths from $(0, 2)$ to $(n-1, n+2)$ that do not cross the main diagonal is equal to the number of $(n+2)$-permutations that avoid 2-13, begin with the pattern 123 and end with the pattern 21.



$$\Psi_k(x) = \frac{\sqrt{3}}{2}x + \frac{\pi}{k}$$

$$\Phi_k(x) = \frac{e^{x/2}}{(k-1)!} \sec(\Psi_6(x)) \int_0^x e^{-t/2} t^{\ell-1} \sin(\Psi_3(t))\ dt$$

$$\Theta_k(x) = \int_0^x \sec(\Psi_6(t)) \left(\sin(\Psi_3(t)) - \frac{\sqrt{3}}{2} e^{-t/2} \right) \left(\Phi_k(t) + \frac{t^{k-1}}{(k-1)!} \right)\ dt$$

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$$\Theta_k(x) = \int_0^x \sec(\Psi_6(t))\left(\sin(\Psi_3(t)) - \frac{\sqrt{3}}{2}e^{-t/2}\right)\left(\Phi_k(t) + \frac{t^{k-1}}{(k-1)!}\right)\,dt$$

Theorem. [Kitaev and Mansour] The EGF $E_{123}^{12\cdots k,12\cdots\ell}(x)$ is

- 0, if $k \geqslant 3$ or $\ell \geqslant 3$;
- $x-\frac{1}{2}-\frac{\sqrt{3}}{2}\tan(\Psi_6(x))+\sec(\Psi_6(x))\left(\frac{\sqrt{3}}{2}\left(e^{x/2}+e^{-x/2}\right)-\sin(\Psi_3(x))\right)$, if $k=2$ and $\ell=2$;
- $\frac{\sqrt{3}}{2}e^{x/2}\sec(\Psi_6(x))-1$, if $k=1$ and $\ell=1$;
- $\frac{\sqrt{3}}{2}e^{x/2}\sec(\Psi_6(x))-\frac{1}{2}-\frac{\sqrt{3}}{2}\tan(\Psi_6(x))$, otherwise.

Theorem. [Elizalde and Noy] Let $h(x) = \sqrt{(x-1)(x+3)}$. Then

$$\sum_{\pi \in S} x^{(123)\pi} \frac{t^{|\pi|}}{|\pi|!} = \frac{2h(x)e^{\frac{1}{2}(h(x)-x+1)t}}{h(x)+x+1+(h(x)-x-1)e^{h(x)t}}$$

$$\sum_{\pi \in S} x^{(213)\pi} \frac{t^{|\pi|}}{|\pi|!} = \frac{1}{1-\int_0^t e^{(x-1)z^2/2}\, dz}$$

Fourth annual conference on

Permutation patterns

Reykjavík University

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The End ;-)